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Planar Geometry Ferrofluid Flows in Spatially Uniform Sinusoidally Time-varying Magnetic Fields

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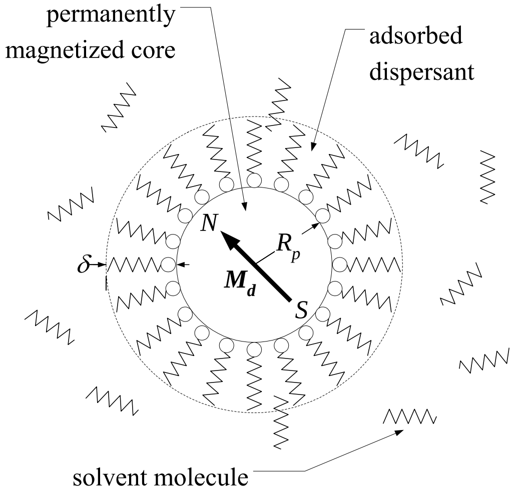
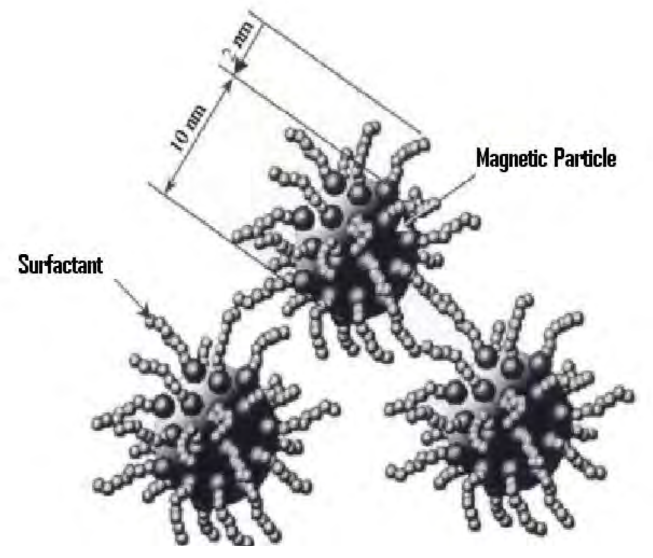
COMSOL CONFERENCE 2011

Ferrofluids

- Ferrofluids
  - Nanosized particles in carrier liquid (diameter~10nm)
  - Super-paramagnetic, single domain particles
  - Coated with a surfactant (~2nm) to prevent agglomeration
- Applications
  - Hermetic seals (hard drives)
  - Magnetic hyperthermia for cancer treatment



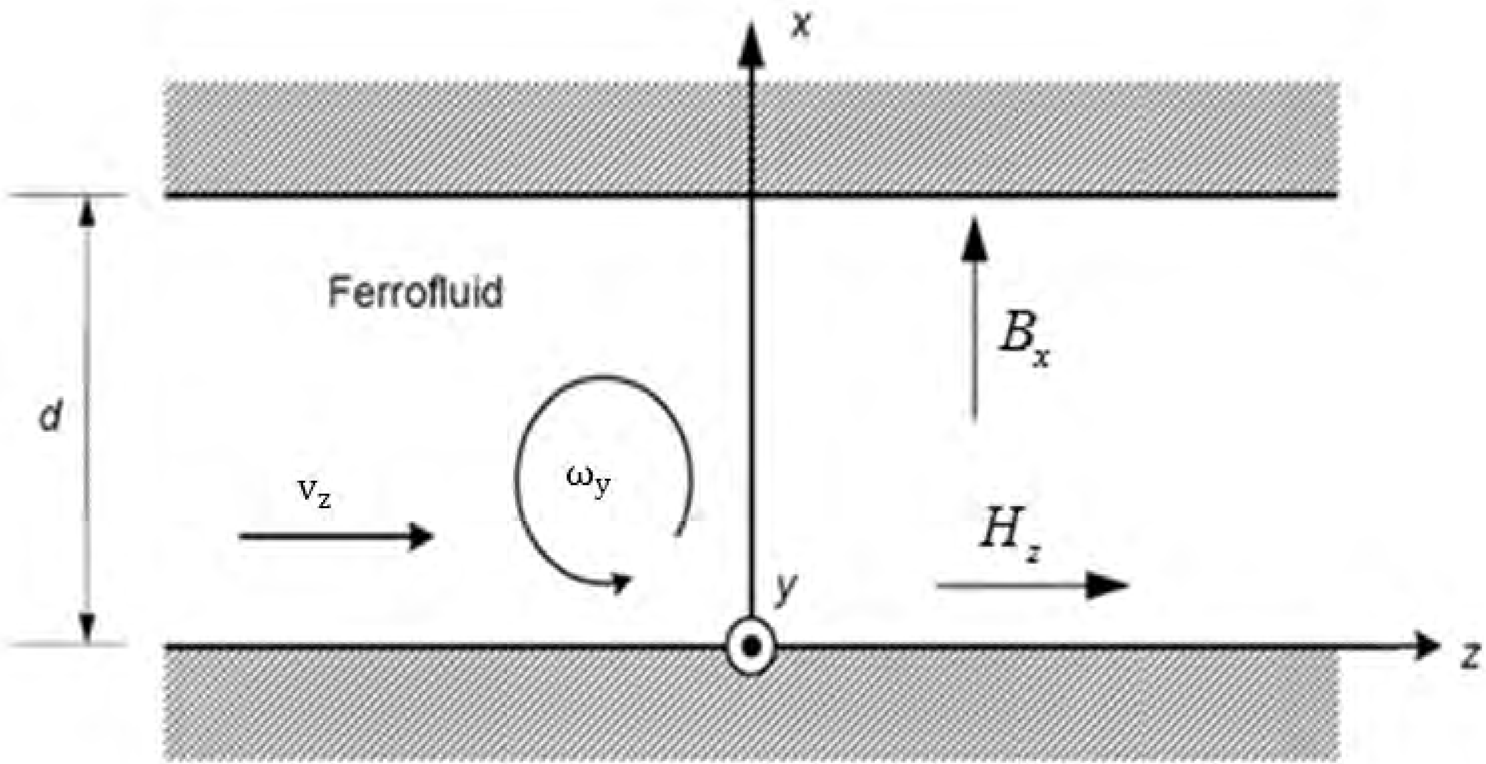


S. Odenbach, *Magnetoviscous Effects in Ferrofluids: Springer, 2002.*

Motivation

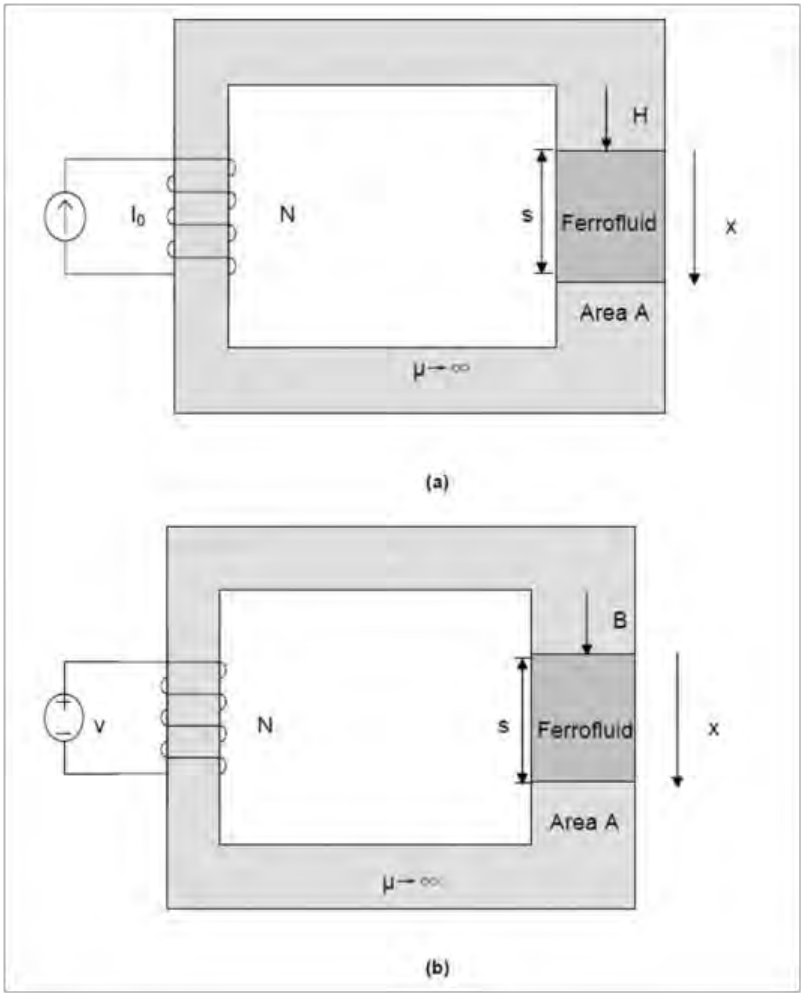
- Prior ferrofluid problems solved in COMSOL are usually in spherical and cylindrical geometries
- Ferrofluid pumping in planar geometry subjected to perpendicular and tangential magnetic fields
  - Well posed problem with analytical solutions
- Traditionally solved using mathematical packages such as Mathematica
  - Can COMSOL replicate these results?

Planar Geometry Setup



$$\mathbf{v} = v_z(x)\mathbf{i}_z, \omega = \omega_y(x)\mathbf{i}_y$$

How to impose B_x field?



(a)

DC Current source gives H=NI/s

(b)

$V=\Lambda_0\delta(t) \rightarrow B=\Lambda_0/A$

X. He, "Ferrohydrodynamic flows in uniform and non-uniform rotating magnetic fields," Ph.D thesis, Dept. of Electrical Engineering and Computer Science, MIT, Cambridge, MA, 2006.
S. Khushrushahi, "Ferrofluid Spin-up Flows in Uniform and Non-uniform Rotating Magnetic Fields," PhD, Dept. of Electrical Engineering and Computer Science, MIT, Cambridge, 2010.

Governing Equations

- Extended Navier-Stokes Equation

=0 =0 *Incompressible flow*

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\bullet\nabla)\mathbf{v}\right]=-\nabla p+\mu_0(\mathbf{M}\bullet\nabla)\mathbf{H}+2\zeta\nabla\times\boldsymbol{\omega}+(\lambda+\eta-\zeta)\nabla(\nabla\bullet\mathbf{v})+(\zeta+\eta)\nabla^2\mathbf{v}-\rho g\mathbf{i_x}$$

Neglecting Inertia

  - Boundary condition on **v,** $\mathbf{v}(r=R_{wall})=0$

- Conservation of internal angular momentum

=0 =0

$$J\left[\frac{\partial \boldsymbol{\omega}}{\partial t}+(\mathbf{v}\bullet\nabla)\boldsymbol{\omega}\right]=\mu_0(\boldsymbol{M}\times\boldsymbol{H})+2\zeta(\nabla\times\mathbf{v}-2\boldsymbol{\omega})+(\lambda'+\eta')\nabla(\nabla\bullet\boldsymbol{\omega})+\eta'\nabla^2\boldsymbol{\omega} \qquad \zeta=\frac{3}{2}\eta\phi$$

Neglecting Inertia

  - Boundary condition on **ω** unless η'=0, $\boldsymbol{\omega}(r=R_{wall})=0$

ρ [kg/m^3] is the ferrofluid mass density, p [N/m^2] is the fluid pressure, ζ [Ns/m^2] is the vortex viscosity, η [Ns/m^2] is the dynamic shear viscosity, λ [Ns/m^2] is the bulk viscosity, ω [s^{-1}] is the spin velocity of the ferrofluid, v is the velocity of the ferrofluid, J [kg/m] is the moment of inertia density, η′ [Ns] is the shear coefficient of spin viscosity and λ′[Ns] is the bulk coefficient of spin viscosity, ϕ[%] is the magnetic particle volume fraction

Magnetic Field Equations

- Maxwell's equations for non-conducting fluid

$$\mathbf{M} = Re\left\{\mathbf{M}e^{-j\Omega t}\right\}, \mathbf{B} = Re\left\{\mathbf{B}e^{-j\Omega t}\right\}, \mathbf{H} = Re\left\{\mathbf{H}e^{-j\Omega t}\right\}$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \frac{dB_x}{dx} = 0 \rightarrow B_x = constant$$

$$\nabla \times \mathbf{H} = 0 \rightarrow \frac{dH_z}{dx} = 0 \rightarrow H_z = constant$$

$$\mathbf{B} = \mu_0\left(\mathbf{H} + \mathbf{M}\right)$$

- Assumption

$$\mathbf{M}_{\text{eq}} = \chi \mathbf{H}_{\text{fluid}}$$

- Magnetic Relaxation Equation

$$\frac{\partial \boldsymbol{M}}{\partial t} + \mathbf{v} \bullet \nabla \boldsymbol{M} - \boldsymbol{\omega} \times \boldsymbol{M} + \frac{1}{\tau_{eff}}(\boldsymbol{M} - \boldsymbol{M}_0) = 0$$

- Langevin Equation

$$\boldsymbol{M}_0 = \boldsymbol{M}_s[\coth(a) - \frac{1}{a}], a = \frac{\mu_0 H_0 M_d V_p}{kT}$$

$$\frac{1}{\tau_{eff}} = \frac{1}{\tau_B} + \frac{1}{\tau_N} \quad \tau_B = 3V_h\frac{\eta_0}{kT}, \tau_N = \frac{1}{f_0}exp\left(\frac{K_a V_p}{kT}\right)$$

$\mathbf{M_s}$ [Amps/m] represents the saturation magnetization of the material, $\mathbf{M_d}$ [Amps/m] is the domain magnetization (446kA/m for magnetite), V_h is the hydrodynamic volume of the particle, V_p is the magnetic core volume per particle, T is the absolute temperature in Kelvin, $k = 1.38 \times 10^{-23}$ [J/K] is Boltzmann's constant, f_0 [1/s] is the characteristic frequency of the material and K_a is the anisotropy constant of the magnetic domains

Substituting in Relaxation Equation

$$j\Omega M_x - \omega_y M_z + \frac{M_x}{\tau} = \frac{\chi_0}{\tau} H_x$$

$$j\Omega M_z + \omega_y M_x + \frac{M_z}{\tau} = \frac{\chi_0}{\tau} H_z$$

$$B = Re[B_x \mathbf{i_x} + B_z(x)\mathbf{i_z}]e^{(-j\Omega t)}$$

$$H = Re[H_x(x)\mathbf{i_x} + H_z \mathbf{i_z}]e^{(-j\Omega t)}$$

$$B_x = \mu_0(H_x + M_x) \rightarrow H_x = \frac{B_x}{\mu_0} - M_x$$

$$M_x = \frac{\chi_0\left[H_z\left(\omega_y\tau\right) + (j\Omega\tau + 1)B_x/\mu_0\right]}{\left[\left(\omega_y\tau\right)^2 + \left(j\Omega\tau + 1\right)(j\Omega\tau + 1 + \chi_0)\right]}$$

$$M_z = \frac{\chi_0\left[H_z\left(j\Omega\tau + 1 + \chi_0\right) - B_x\omega_y\tau/\mu_0\right]}{\left[\left(\omega_y\tau\right)^2 + \left(j\Omega\tau + 1\right)(j\Omega\tau + 1 + \chi_0)\right]}$$

Force and Torque Densities

$$< \mathbf{F} >= \frac{\mu_0}{2}\mathrm{Re}\left[\left(\mathbf{M}\square\nabla\right)\mathbf{H}^*\right] \rightarrow \mathrm{F_x} = -\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\mu_0}{4}\left|M_x\right|^2\right), \mathrm{F_z} = 0$$

$$< \mathbf{T} >= \frac{\mu_0}{2}\mathrm{Re}\left[\mathbf{M}\times\mathbf{H}^*\right] \rightarrow \mathrm{T_y} = \frac{1}{2}\mathrm{Re}\left[M_z B_x^* - \mu_0 M_x^*\left(M_z + H_z\right)\right]$$

Linear and Angular Momentum Eqns

$$0 = -\frac{\partial p'}{\partial \mathrm{z}} + 2\zeta\frac{d\omega_y}{dx} + \left(\zeta+\eta\right)\frac{\mathrm{d}^2 v_z}{\mathrm{dx}^2}$$

$$0 = \mathrm{T_y} - 2\zeta\left(\frac{\mathrm{d}v_z}{\mathrm{dx}} + 2\omega_y\right) + \eta'\frac{\mathrm{d}^2\omega_y}{\mathrm{dx}^2}$$

$$p' = p + \frac{\mu_0}{4}\left|M_x\right|^2 + \rho\mathrm{gx}$$

Normalization and Substitution

$$\tilde{\Omega} = \Omega\tau, \tilde{\mathbf{H}} = \frac{\hat{\mathbf{H}}}{H_0}, \tilde{\mathbf{M}} = \frac{\hat{\mathbf{M}}}{H_0}, \tilde{\mathbf{B}} = \frac{\hat{\mathbf{B}}}{\mu_0 H_0}, \tilde{x} = \frac{x}{d}, \tilde{v}_z = \frac{v_z \tau}{d}, \tilde{\omega}_y = \omega_y \tau,$$

$$\tilde{T}_y = \frac{T_y}{\mu_0 H_0^2}, \tilde{\eta} = \frac{2\eta}{\mu_0 H_0^2 \tau}, \tilde{\eta}' = \frac{\eta'}{\mu_0 H_0^2 \tau d^2}, \tilde{\zeta} = \frac{2\zeta}{\mu_0 H_0^2 \tau}, \frac{\partial \tilde{p}'}{\partial \tilde{z}} = \frac{d}{\mu_0 H_0^2} \frac{\partial p'}{\partial z}$$

$$\tilde{M}_x = \frac{\chi_0 \left[\tilde{\omega}_y \tilde{H}_z + \left(j\tilde{\Omega} + 1 \right) \tilde{B}_x \right]}{\left[\tilde{\omega}_y{}^2 + \left(j\tilde{\Omega} + 1 \right)\left(j\tilde{\Omega} + 1 + \chi_0 \right) \right]}$$

$$\tilde{M}_z = \frac{\chi_0 \left[\left(j\tilde{\Omega} + 1 + \chi_0 \right) \tilde{H}_z - \tilde{B}_x \tilde{\omega}_y \right]}{\left[\tilde{\omega}_y^2 + \left(j\tilde{\Omega} + 1 \right)\left(j\tilde{\Omega} + 1 + \chi_0 \right) \right]}$$

$$< \tilde{T}_y > = \frac{1}{2} \mathrm{Re}\left[\tilde{M}_z \tilde{B}_x^* - \tilde{M}_x^* \left(\tilde{H}_z + \tilde{M}_z \right) \right]$$

$$0 = -\frac{\partial \tilde{p}'}{\partial \tilde{z}} + \tilde{\zeta} \left(\frac{d\tilde{\omega}_y}{d\tilde{x}} \right) + \frac{1}{2} \left(\tilde{\zeta} + \tilde{\eta} \right) \frac{d^2 \tilde{v}_z}{d\tilde{x}^2}$$

$$< \tilde{T}_y > - \tilde{\zeta} \left(\frac{d\tilde{v}_z}{d\tilde{x}} + 2\tilde{\omega}_y \right) + \tilde{\eta}' \frac{d^2 \tilde{\omega}_y}{d\tilde{x}^2} = 0$$

Torque Density

- Analytical Torque Density

$$\langle\tilde{T}_y\rangle = \frac{\chi_0}{2}\Big[-\tilde{\omega}_y\Big[|\tilde{B}_x|^2\big(\tilde{\omega}_y^2 - \tilde{\Omega}^2 + 1\big) + |\tilde{H}_z|^2\big[\tilde{\omega}_y^2 - \tilde{\Omega}^2 + (1+\chi_0)^2\big]\Big] + 2\Re\Big[\big[\chi_0\big(\tilde{\omega}_y^2 - \tilde{\Omega}^2\big) + \mathrm{i}\tilde{\Omega}\big(\tilde{\omega}_y^2 - \tilde{\Omega}^2 - 1 - \chi_0\big)\big]\big[H_z B_x^*\big]\Big]\Big] \Big/ \Big[\big[\tilde{\omega}_y^2 - \tilde{\Omega}^2 + 1 + \chi_0\big]^2 + (2+\chi_0)^2\tilde{\Omega}^2\Big].$$

- Small spin limit Torque Density

$$\lim_{\tilde{\omega}_y \ll 1} < \tilde{T}_y >= \tilde{T}_0 + \alpha\tilde{\omega}_y$$

$$\tilde{T}_0 = -\frac{\chi_0 \operatorname{Re}\Big[\big[\chi_0\tilde{\Omega}^2 + j\tilde{\Omega}\big(\tilde{\Omega}^2 + 1 + \chi_0\big)\big]\big[\tilde{H}_z\tilde{B}_x^*\big]\Big]}{\big[1+\chi_0+\tilde{\Omega}^2\big]^2 + \chi_0^2\tilde{\Omega}^2}$$

$$\alpha = \frac{\chi_0}{2}\frac{\Big[\big|\tilde{B}_x\big|^2\big(\tilde{\Omega}^2 - 1\big) + \big|\tilde{H}_z\big|^2\big[\tilde{\Omega}^2 - (1+\chi_0)^2\big]\Big]}{\big[1+\chi_0+\tilde{\Omega}^2\big]^2 + \chi_0^2\tilde{\Omega}^2}$$

COMSOL Setup

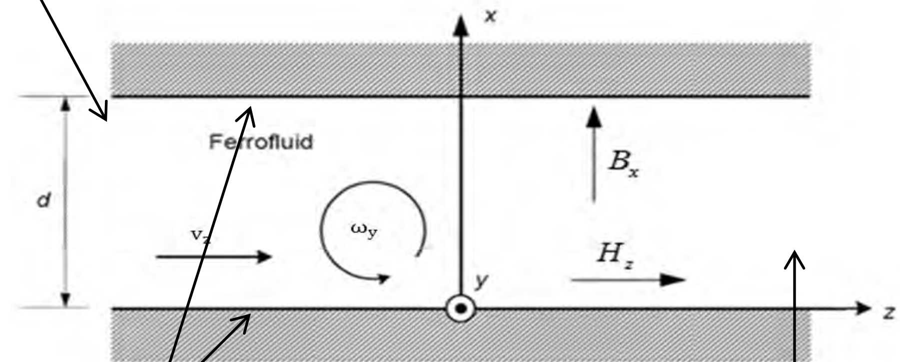
- Linear Momentum Equation
  - 2D Incompressible Navier Stokes Module

$$0 = -\frac{\partial \tilde{p}'}{\partial \tilde{z}} + \tilde{\zeta}\left(\frac{d\tilde{\omega}_y}{d\tilde{x}}\right) + \frac{1}{2}\left(\tilde{\zeta}+\tilde{\eta}\right)\frac{d^2\tilde{v}_z}{d\tilde{x}^2}$$

COMSOL Subdomain quantities	Value
ρ	0
η	$\frac{1}{2}\left(\tilde{\zeta}+\tilde{\eta}\right)$
Fx,Fy	$\tilde{\zeta}\left(\frac{d\tilde{\omega}_y}{d\tilde{x}}\right)$

Inlet BC – Pressure, No viscous Stress

$$p_o = -\tilde{p}', \frac{\partial \tilde{p}'}{\partial z} > 0$$



No Slip BC

$\tilde{v}_z = 0$

Outlet BC, Normal Stress

$f_0=0$

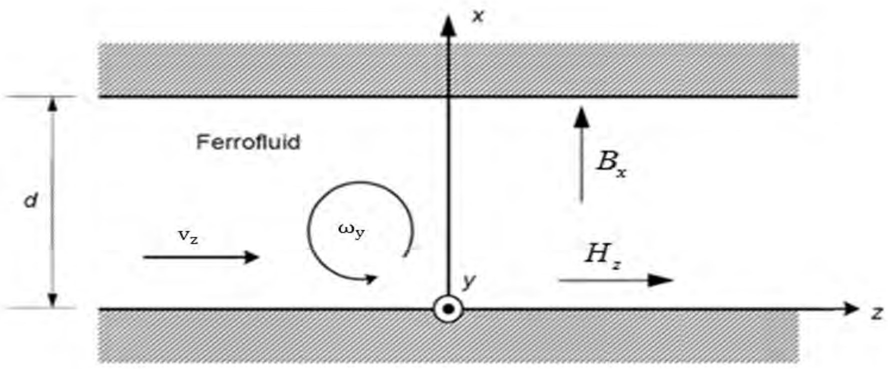
COMSOL Setup

- Angular Momentum Equation
  - General PDE Equation

$$<\tilde{T}_y> -\tilde{\zeta}\left(\frac{d\tilde{v}_z}{d\tilde{x}}+2\tilde{\omega}_y\right)+\tilde{\eta}'\frac{d^2\tilde{\omega}_y}{d\tilde{x}^2}=0$$

COMSOL Subdomain quantities	Value
Γ	0,0
F	$<\tilde{T}_y> -\tilde{\zeta}\left(\frac{d\tilde{v}_z}{d\tilde{x}}+2\tilde{\omega}_y\right)+\tilde{\eta}'\frac{d^2\tilde{\omega}_y}{d\tilde{x}^2}$
e_a,d_a	0,0

Boundary Conditions	COMSOL Quantities
All walls (if $\tilde{\eta}' \neq 0$)	Dirichlet boundary condition R= -$\tilde{\omega}_y$, G=0
All walls (if $\tilde{\eta}' = 0$)	Neumann boundary condition G=0

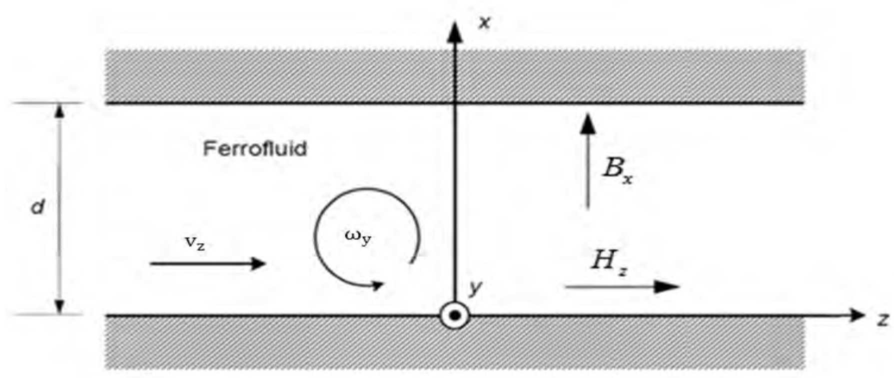


COMSOL Setup

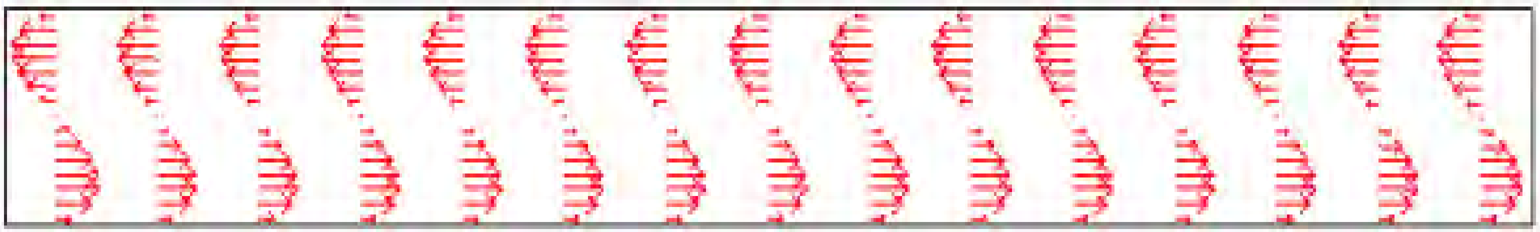
- Magnetic Relaxation Equation
  - 2D Perpendicular Induction Currents, Vector Potential

Boundary Conditions	COMSOL Quantities
All walls	H_0 = $\tilde{H}_z, \tilde{H}_x = \tilde{B}_x - \tilde{M}_x$

COMSOL Subdomain quantities	Value
M	$\frac{\chi_0\left[\tilde{\omega}_y \tilde{H}_z + \left(j\tilde{\Omega}+1\right)\tilde{B}_x\right]}{\left[\tilde{\omega}_y{}^2 + \left(j\tilde{\Omega}+1\right)\left(j\tilde{\Omega}+1+\chi_0\right)\right]}, \frac{\chi_0\left[\left(j\tilde{\Omega}+1+\chi_0\right)\tilde{H}_z - \tilde{B}_x\tilde{\omega}_y\right]}{\left[\tilde{\omega}_y^2 + \left(j\tilde{\Omega}+1\right)\left(j\tilde{\Omega}+1+\chi_0\right)\right]}$



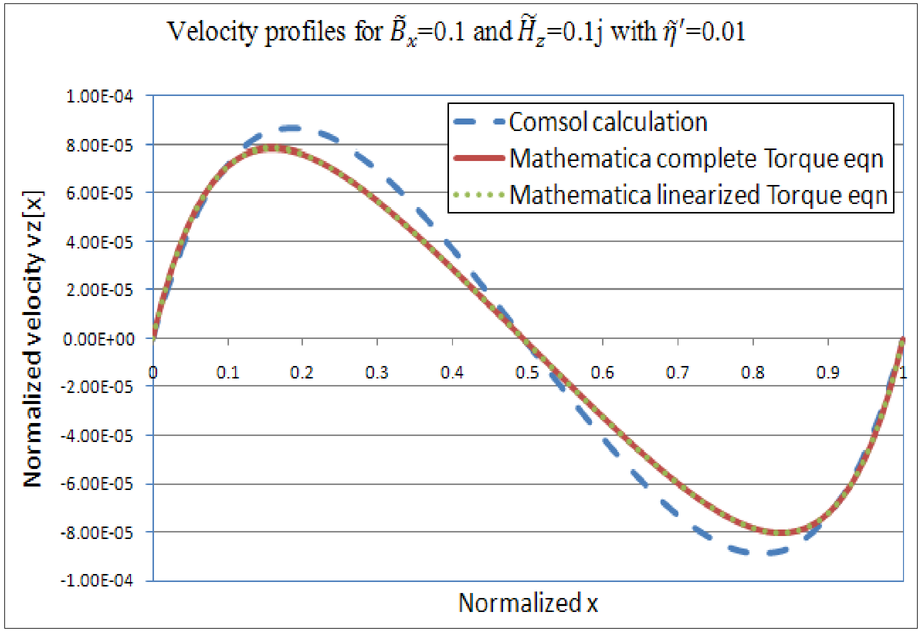
η’≠0 Results, Weak Rotating Fields

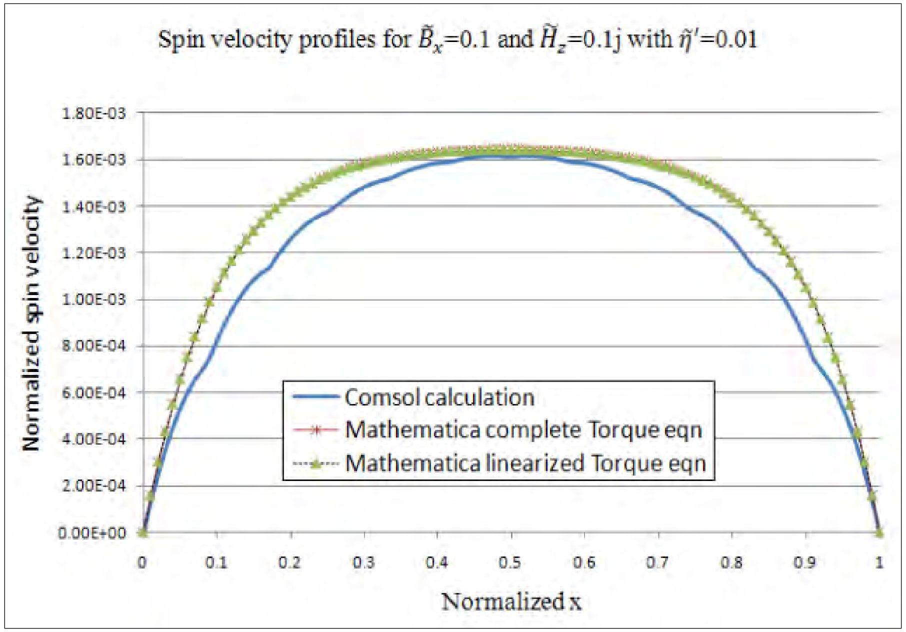


Parameters used – $\chi_0 = 1, \tilde{\eta} = 1, \tilde{\zeta} = 1, \frac{\partial \tilde{p}'}{\partial \tilde{z}} = 0.00001, \tilde{\Omega} = 1, \tilde{\eta}' = 0.01$

S. Khushrushahi, "Ferrofluid Spin-up Flows in Uniform and Non-uniform Rotating Magnetic Fields," PhD, Dept. of Electrical Engineering and Computer Science, MT, Cambridge, 2010.

η'≠0 Results, Weak Rotating Fields

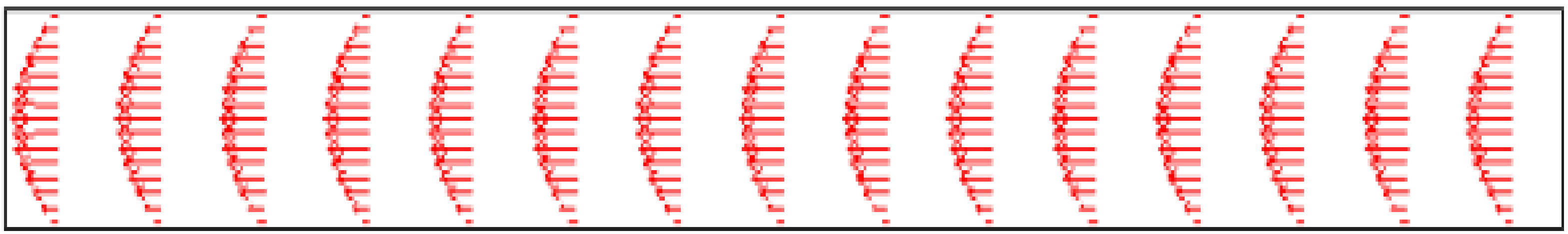




Parameters used – $\chi_0 = 1, \tilde{\eta} = 1, \tilde{\zeta} = 1, \dfrac{\partial \tilde{p}'}{\partial \tilde{z}} = 0.00001, \tilde{\Omega} = 1, \tilde{\eta}' = 0.01$

S. Khushrushahi, "Ferrofluid Spin-up Flows in Uniform and Non-uniform Rotating Magnetic Fields," PhD, Dept. of Electrical Engineering and Computer Science, MT, Cambridge, 2010.

η'=0 Results, Strong Rotating Fields

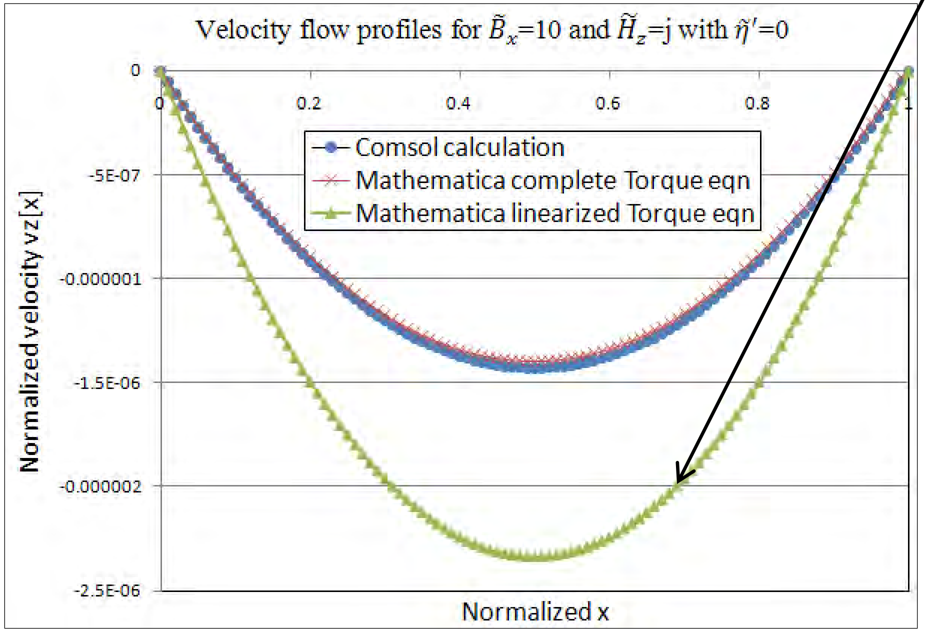


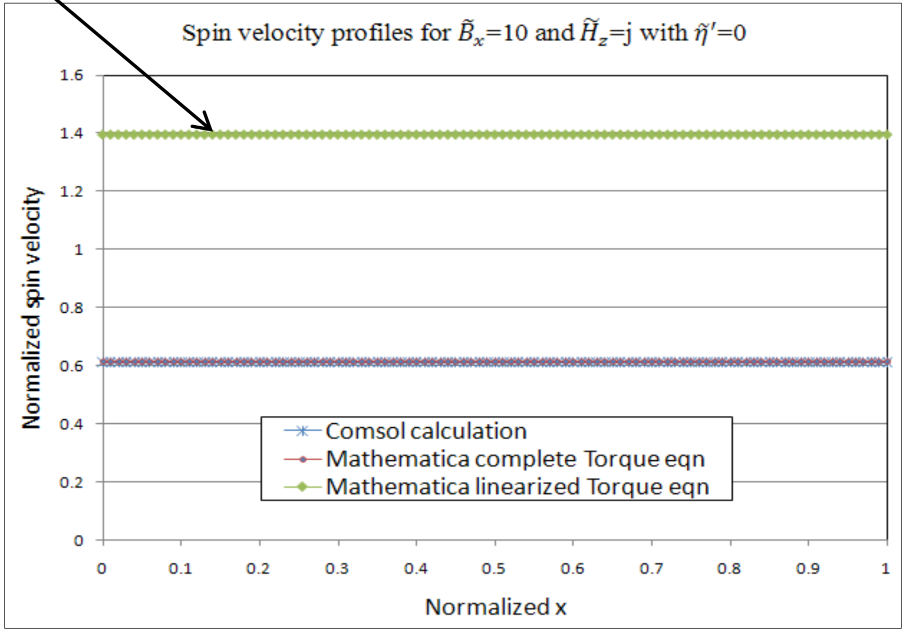
Parameters used – $\chi_0 = 1, \tilde{\eta} = 1, \tilde{\zeta} = 1, \frac{\partial \tilde{p}'}{\partial \tilde{z}} = 0.00001, \tilde{\Omega} = 1, \tilde{\eta}' = 0$

S. Khushrushahi, "Ferrofluid Spin-up Flows in Uniform and Non-uniform Rotating Magnetic Fields," PhD, Dept. of Electrical Engineering and Computer Science, MT, Cambridge, 2010.

η'=0 Results, Strong Rotating Fields

Small Spin Velocity limit does not hold

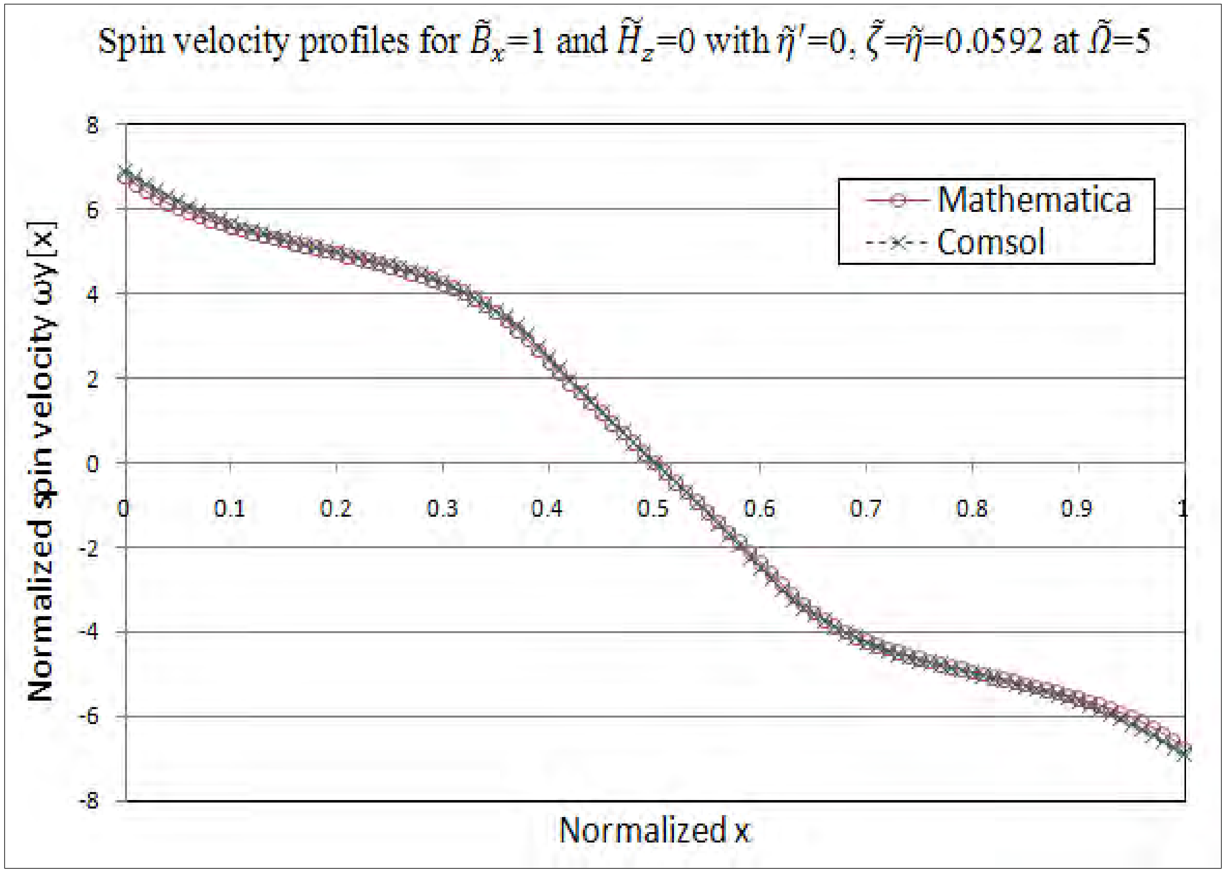




Parameters used – $\chi_0 = 1, \tilde{\eta} = 1, \tilde{\zeta} = 1, \dfrac{\partial \tilde{p}'}{\partial \tilde{z}} = 0.00001, \tilde{\Omega} = 1, \tilde{\eta}' = 0$

S. Khushrushahi, "Ferrofluid Spin-up Flows in Uniform and Non-uniform Rotating Magnetic Fields," PhD, Dept. of Electrical Engineering and Computer Science, MT, Cambridge, 2010.

“Kinks” for special parameters



Parameters used – $\chi_0 = 1, \tilde{\eta} = \tilde{\zeta} = 0.0592, \dfrac{\partial \tilde{p}'}{\partial \tilde{z}} = 1, \tilde{\Omega} = 5, \tilde{\eta}' = 0$

L. L. V. Pioch, "Ferrofluid flow & spin profiles for positive and negative effective viscosities," Masters of Engineering, Dept. of Electrical Engineering and Computer Science, MIT, 1997.
M. Zahn and L. Pioch, "Ferrofluid flows in AC and traveling wave magnetic fields with effective positive, zero or negative dynamic viscosity," *J. Magn. Magn. Mater., vol. 201, p. 144, 1999.*
M. Zahn and L. L. Pioch, "Magnetizable fluid behaviour with effective positive, zero or negative dynamic viscosity," *Indian Journal of Engineering & Materials Sciences, vol. 5, pp. 400-410, 1998.*

Conclusions

- Ferrohydrodynamic flows are difficult to model
  - Coupling of five vector equations
    - Linear and angular momentum equations
    - Gauss's law for magnetic flux density
    - Ampere's law with no free current
    - Ferrofluid magnetic relaxation equation
- Solving the basic planar geometry ferrofluid pumping problem is valuable before moving to cylindrical and spherical geometries
- COMSOL gives identical results to prior software of choice - Mathematica