

# Pumping Tests: Comparison of 2D Numerical and 1D Analytical Solutions

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## Abstract:

Pumping tests are the major field experiments within the toolbox of a hydrogeologist. The evaluation of the tests depends on specific and local groundwater conditions. Mostly these cannot be considered by common evaluation methods, which are based on 1D analytical solutions. 2D numerical models offer the chance to overcome this obstacle. Simulations using COMSOL Multiphysics are set up and performed for fully and partially penetrating wells in confined, leaky and unconfined aquifers. We compare numerical with analytical solutions of Theis and Hantush. The simulations can easily be extended to include anisotropy, general boundary conditions, skin zone effects etc. In connection with an optimization module such simulation models may become a more flexible and reliable tool for hydrogeologists.

**Keywords:** Aquifer, Pumping test, Analytical Solutions

## 1. Introduction

In a pumping test the drawdown of the groundwater water table is recorded in response to the start of operating a pump in a well. The drawdown delivers clue data for the determination of characteristic aquifer properties. For hydrogeologists pumping tests are the most common method to obtain values for the transmissivity and storativity of an aquifer (Kruseman & de Ridder 1990). For the evaluation of pumping tests it is common practice to compare field measurements with 1D analytical solutions of the relevant differential equation. The 1D solutions are derived assuming idealized conditions. Numerical methods offer the chance to check how far 1D solutions fail if assumptions for local conditions and 1-dimensionality are not fulfilled.

Major aim of the study is to find the error of the analytical solutions in cases of non-ideal situations, i.e. when the well-screen is not penetrating the entire aquifer. Moreover we deal with times immediately after pumping start, with short distances between pumping and observation wells and with unconfined aquifers. The results of the comparison provide important information, under which conditions the common evaluation methods are appropriate and when they are not.

Yeh & Chang (2013) give an overview on recent advances in the modelling of well hydraulics. Huang *et al.* (2015) provide a categorization of solutions for the constant-flux pumping in confined aquifers. They describe analytical as well as numerical approaches. Our numerical model represents a 2D vertical cross-section of the aquifer. We are using 2D cylinder coordinates  $r$  (radial) and  $z$  (vertical) with the centre of the pumping well as symmetry axis. The differential equation for hydraulic head  $h$  in a vertical cross-section of an aquifer is given by

$$S_s \frac{\partial h}{\partial t} = K_h \left( \frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} \right) + K_v \frac{\partial^2 h}{\partial z^2} \quad (1)$$

(Yeh & Chang 2013), where  $S_s$  denotes the specific storativity,  $K_h$  the horizontal and  $K_v$  the vertical hydraulic conductivity. Equation (1) is derived from the fluid mass conservation principle and Darcy's Law for porous media flow. If the variable of hydraulic head  $h$  is replaced by drawdown  $s=h_0-h$ , where  $h_0$  denotes the initial state, one obtains:

$$S_s \frac{\partial s}{\partial t} = K_h \left( \frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} \right) + K_v \frac{\partial^2 s}{\partial z^2} \quad (2)$$

(Chang *et al.* 2011). In case of an unconfined aquifer equation (1) is valid for  $z < h(r,t)$ , if  $h$  is measured with reference to the base of the aquifer. This formulation is utilized in the set-up of the model for the unconfined case with a changing model region due to a free boundary on the top.

For a homogeneous isotropic aquifer with  $K = K_h = K_v$  and constant thickness  $H$  equation (2) can be modified to:

$$S \frac{\partial s}{\partial t} = T \left( \frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial z^2} \right) \quad (3)$$

where  $T = K \cdot H$  denotes aquifer transmissivity and  $S = S_s \cdot H$  the storativity of the aquifer. For the derivation of an 1D analytical solutions it is assumed that the vertical velocity components and thus the corresponding gradients in equation (3) can be neglected. Then the solution

$$s(r,t) = \frac{Q}{4\pi T} W\left(\frac{Sr^2}{4Tt}\right) \quad (4)$$

describes drawdown in the vicinity of an ideal well in a confined aquifer.  $Q$  denotes the pumping rate. Equation (4) is mostly referred to as Theis solution (Theis 1937). It describes drawdown  $s$  as a function of the radial distance to the well centre-line  $r$  and time  $t$  and is expressed by the function  $W(u)$ , which among hydrogeologists is known as *well-function*. In mathematics it is referred to as exponential integral, defined by:

$$W(u) = E_1(u) = \int_u^{\infty} \frac{\exp(-\zeta)}{\zeta} d\zeta \quad (5)$$

The variable  $u = Sr^2 / 4Tt$  combines both space and time variables with the parameters storativity  $S$  and the transmissivity  $T$ . Using equation (5) the derivative of solution (4) can be evaluated easily, leading to the proportionality relation

$$\frac{\partial s}{\partial r} \propto \exp\left(-\frac{Sr^2}{4Tt}\right) \quad (6)$$

According to relation (6) the gradient  $\partial s / \partial r$  that is related to the outflow velocity and the pumping rate, vanishes at time  $t=0$ . The pumping rate represented by solution (4) is thus not a constant, but rises gradually from zero to a final finite value. For times immediate after pump start we can thus not expect a correspondence between solution (4) and a real pumping test, which operates with a constant pumping rate from the start. Also a numerical solution for a test with constant pumping rate from the very start will not reproduce the 1D analytical solution for small simulation times.

For semi-confined leaky aquifers the formula (4) has to be extended:

$$s(r,t) = \frac{Q}{4\pi T} W\left(\frac{Sr^2}{4Tt}, \frac{r}{\lambda}\right) \quad (7)$$

with  $\lambda = \sqrt{Tc}$ , where  $c$  denotes the leakance time of the overlying semi-permeable layer (Hantush 1964), and  $W(u,\mu)$  the Hantush wellfunction, defined by:

$$W(u,\mu) = \int_u^{\infty} \frac{1}{\zeta} \exp\left(-\zeta - \frac{\mu^2}{4\zeta}\right) d\zeta \quad (8)$$

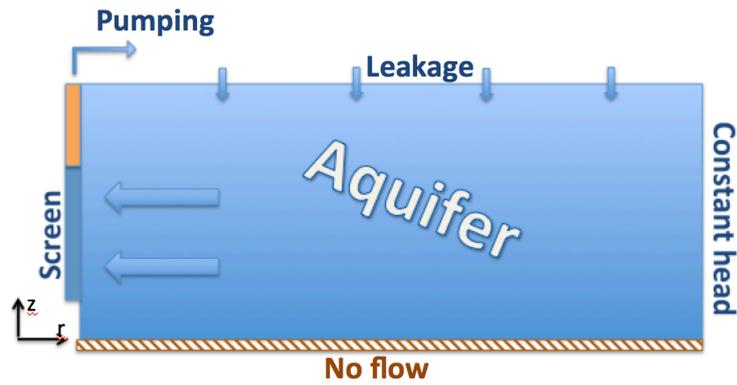
Formula (7) was derived using a source term in the differential equation (3) (Hantush & Jacob 1955). Using the more realistic boundary condition at the top of the modelled cross-section Hantush (1967) showed conditions, under which solution (7) can be taken as a valid approximation. In the numerical evaluation of equation (8) I follow Maas & Veling (2010). The proposed numerical computation of the Hantush well-function is implemented as a MATLAB function that is called by COMSOL Multiphysics via the direct link.

In case of partial penetration the behaviour of the aquifer will differ from the analytical solutions, if 2D effects come into play. More specifically: vertical velocity components, that are assumed to be zero in the derivation of the analytical solutions, may not be negligible anymore. The deviation from the analytical solution is most severe in the vicinity of the well. It is commonly accepted that the radial distance from the well should be larger than half of the aquifer thickness (Kono & Nishigaki 1978). However, this condition may not be fulfilled in many pumping tests, as the observations are placed near the pumping well in order to observe significant drawdown in short times. For that reason I examine observation points near to the pumping well.

## 2. Model Set-up

In COMSOL we use the Darcy mode for porous media flow. Note that in COMSOL the pressure is used as dependent variable instead of the hydraulic head. In case of an unconfined aquifer the model is complemented by the arbitrary Lagrangian Eulerian mode in order to account for the changing water table. The free boundary changes due to the moving mesh. A corresponding approach was presented by Jin *et al.* (2012, 2014).

The initial model region is a rectangle with aquifer thickness as height  $H$  and the reach of the well as length  $L$ . The sketch in Figure 1 illustrates the set-up and boundary conditions. A filter that extends from the top to the bottom of the aquifer characterizes an ideal well. Most wells are not ideal or it is not known, if they are ideal or not. One aim here is to compare the flow regime in the aquifer near ideal and non-ideal wells.



**Figure 1.** Schematic illustration of 2D model region and boundary conditions

The model concerns flow within the aquifer only, not in the borehole itself. The vertical boundary on the left is thus located at the wall of the well. The connection between aquifer and well is given at the filter, where we assume a constant mass pumping rate  $N_0$ . For constant head pumping tests a constant head boundary has to be prescribed (Chang *et al.* 2011). An impermeable layer underlies the aquifer, so that no flow conditions can be assumed at the bottom. At the upper boundary leakage is represented by a Cauchy boundary condition with external hydraulic head  $h_0$  and conductance  $1/c$ , where  $c$  denotes leakance time. In case of zero leakage this is identical to the no flow condition. At the right boundary a Dirichlet condition is specified, representing no vertical fluxes. For the unconfined situation we allow changes of the boundary location at the top of the model and on the left at the well boundary above the filter. At both boundaries only vertical movements are allowed. On the left side these are realized by roller boundary conditions. Table 1 lists all reference case parameters for the confined and leaky cases.

**Table 1.** Parameters of the reference case

Name	Expression	Value	Description
$r_0$	0.15[m]	0.15 m	well radius
$L$	300[m]	300 m	reach
$h_0$	52[m]	52 m	initial head
$H$	50[m]	50 m	aquifer thickness
$K$	$3 \cdot 10^{-4}$ [m/s]	3E-4 m/s	hydraulic conductivity
$T$	$K \cdot H$	0.015 m <sup>2</sup> /s	transmissivity
$\theta$	0.25	0.25	porosity
$\rho$	1000[kg/m <sup>3</sup> ]	1000 kg/m <sup>3</sup>	water density
$S_p$	$0.5 \cdot 10^{-7}$ [1/Pa]	5E-8 1/Pa	compressibility
$S_s$	$S_p \cdot g \cdot \rho$	4.9033E-4 1/m	specific storativity
$S$	$S_s \cdot H$	0.024517	storativity
$D$	$K/S_s$	0.61183 m <sup>2</sup> /s	diffusivity
$c^{\S}$	$10^7$ [s]	1E7 s	deGlee leakance time
$\lambda^{\S}$	$\text{sqrt}(T \cdot c)$	387.3 m	leakance
$b_f$	0, 25, 37.5[m]	0, 25, 37.5 m	filter bottom (ideal well and variants)
$t_f$	50[m]	50 m	filter top
$Q$	0.02[m <sup>3</sup> /s]	0.02 m <sup>3</sup> /s	pumping rate
$q$	$-Q/2/\pi/r_0/(t_f - b_f)$	-5.3065E-4 m/s	pumping velocity (ideal well)
$N_0$	$q \cdot \rho$	-0.53065 kg/(m <sup>2</sup> ·s)	mass pumping rate
$z_o$	$b_f + (t_f - b_f)/2$	25 m	observation well depth (ideal well)
$r_{o2}$	$r_0 + 10$ [m]	10.15 m	observation well 1 distance
$r_{o3}$	$r_{o2} + 10$ [m]	20.15 m	observation well 2 distance

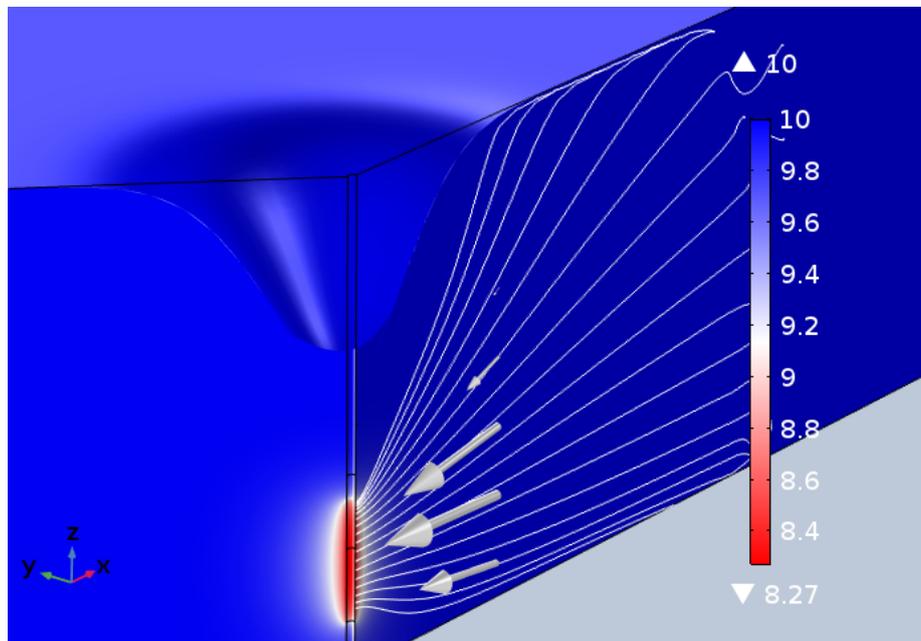
<sup>§</sup> for a leaky aquifer

The model is based on equation (3) for the confined case, and on equation (2) for the unconfined case. The model region extends in horizontal direction from the well wall at position  $r_0$  to the reach of the well  $L$ . In vertical direction the origin is located at the base of the aquifer. The initial state is characterized by a constant head  $h_0$  and zero velocity. Starting from this steady state the model simulates a 48 h pumping period.  $g$  denotes the acceleration due to gravity. Three observation points are specified at distances  $r_0$ ,  $r_{o2}$  and  $r_{o3}$  from the origin. The depth  $z_o$  of all observation points is the same.

For the finite element discretization I chose quadratic elements on an irregular triangular mesh. The mesh is refined at the well screen, restricting the maximum element size to 10 cm.

### 3. Results

Figure 2 provides a 3D illustration of a typical result of a model run for the unconfined aquifer and a not ideal well. The well screen is located at the lower part of the aquifer. The colours indicate values of hydraulic head  $h$  at a given time instant after the start of the pumping test. The colours, described by the colorbar, show the rapid decrease of hydraulic head in the direct vicinity of the well screen. In the depicted state  $h$  decreases from the initial value of 10 m to a minimum of 8.27 m at the well screen. In the figure the upper surface represents the water table. The drawdown cone of the water table is clearly visible above the well screen. Selected streamlines are depicted in white. 3D arrows indicate the velocity direction. The size of the arrows is proportional to the absolute value of the velocity. As expected the velocity field has a maximum at the centre of the well screen.



**Figure 2.** 3D representation of results of a 2D numerical model of an unconfined aquifer; depicted are cone of depression, distribution of hydraulic head, streamlines and velocity arrows

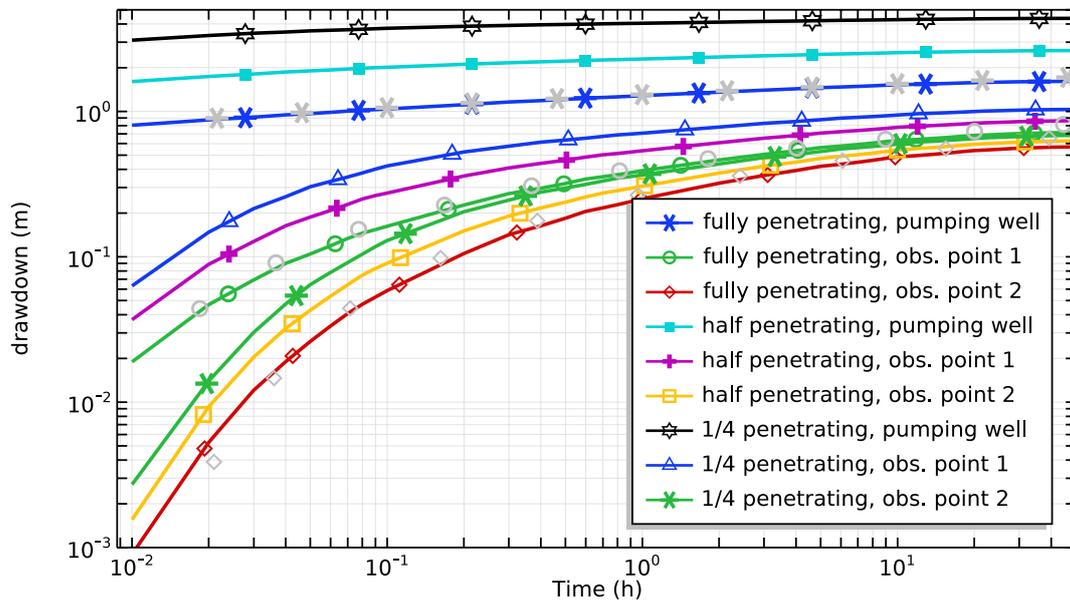
In the following results are presented for the confined, leaky and unconfined aquifers, and parameters from Table 1. For each of the three cases a sensitivity study is performed using parametric sweeps with variations of the well screen bottom  $b_f$  as parameter.  $b_f$  takes the values shown in Table 1. The three values represent an ideal well and two not ideal wells with filter lengths of half and quarter of the initial aquifer thickness.

#### 3.1 Confined Aquifer

Figure 3 depicts the drawdown in the pumping well and at both observation wells as function of time for the reference case. The figure shows the drawdown time series of three model runs of the parametric sweep at three different locations. One location is in the pumping well itself; the others are located at positions  $r_{o2}$  and  $r_{o3}$  at distances 10 and 20 m away from the borehole. In addition grey markers indicate the analytical solution according to formula (5). The shown cases have the same pumping rate; thus for all of the three well constellations there is only one analytical solution.

For the fully penetrating case the analytical and numerical solutions are almost identical. Small deviations can be observed for small simulation times, most pronounced in observation point 2, which is furthest away from the pumping borehole. These deviances result from the inaccuracy of the analytical solution for small times, as mentioned above. However, for the partially penetrating cases the drawdown is significantly higher than for the

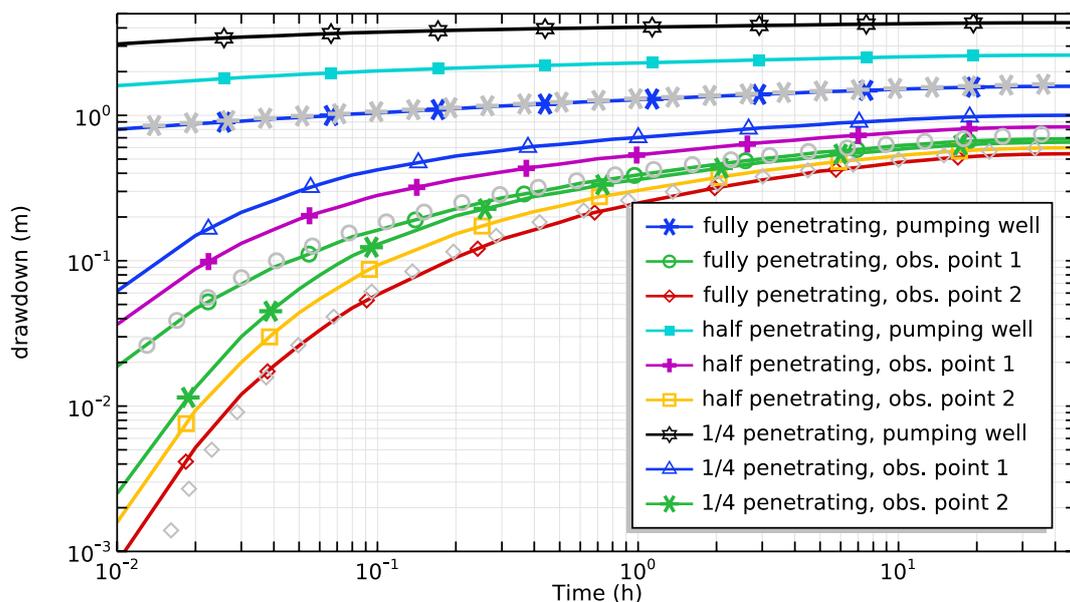
fully penetrating case. If the well screen measures only a quarter of the aquifer thickness, the drawdown within the well is approximately 4 times higher than for the ideal well. The analytical solution is in no way representing the drawdown anymore. In absolute values the deviation between 1D and 2D solutions decrease with distance from the well. The relative error remains the same, as the drawdown also decreases with radial distance.



**Figure 3.** Drawdown as function of time in the pumping well and two observation points for fully, half and  $\frac{1}{4}$  penetrating wells in a confined aquifer; 2D numerical solution in colour with markers, 1D analytical solution (Theis) indicated by grey markers

### 3.2 Leaky Aquifer

For the leaky aquifer a Neumann boundary condition is introduced at the top of the model region. The model is run with the parameters of the reference case, given in Table 1. For three values of filter bottom the drawdown at the three locations in the pumping and observation wells is depicted in Figure 4. Grey markers indicate the corresponding analytical solution according to formula (7). The cases shown have the same pumping rate; thus there is only one analytical solution for all well constellations.



**Figure 4.** Drawdown as function of time in the pumping well and two observation points for fully, half and  $\frac{1}{4}$  penetrating wells in a leaky aquifer; 2D numerical solution in colour with markers, 1D analytical solution (Hantush) indicated by grey markers

As for the confined aquifer for the fully penetrating well numerical and analytical results match with the same exceptions for small simulation times and far distances of the observation well. Again, for the partially

penetrating cases the drawdown is significantly higher than for the fully penetrating case. If the well screen measures a quarter of the aquifer thickness, the drawdown within the well is approximately 4 times higher than for the ideal well. As in the confined case, the analytical solution is in no way representing the drawdown anymore, if the well is not ideal.

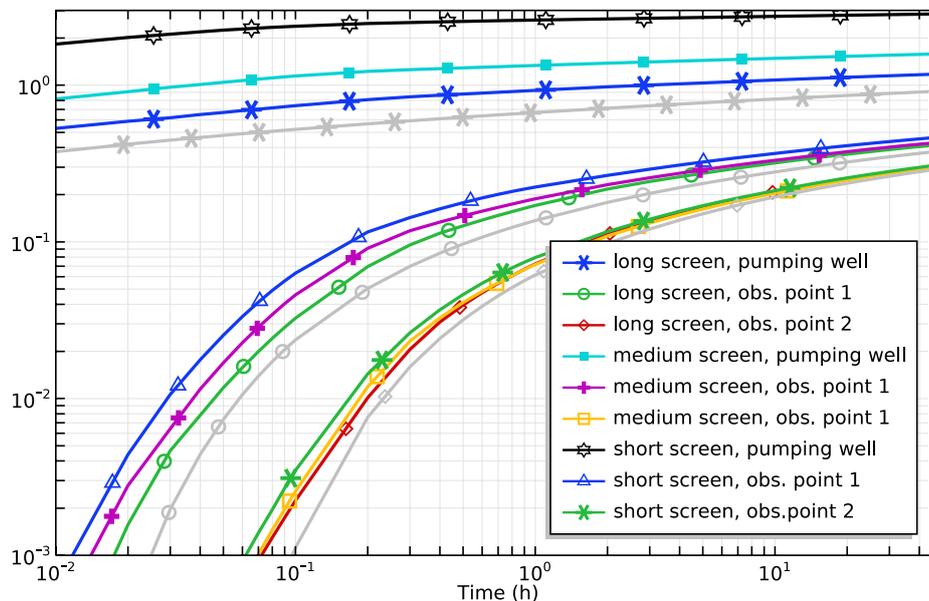
### 3.3 Unconfined Aquifer

For the unconfined case some parameters from the reference case had to be adjusted. As there is a varying aquifer thickness, transmissivity is not defined, and a hydraulic conductivity  $K$  has to be used instead of a transmissivity and the specific  $S_s$  instead of the storativity. The model is based on differential equation (2).

For three values of filter bottom used in a parametric sweep Figure 5 depicts the drawdown at the three locations in the pumping and observation wells. Grey markers indicate the corresponding analytical solution for the confined case according to formula (4). The cases shown have the same pumping rate; thus there is only one analytical solution for each well constellation.

As the analytical solution is derived for the confined case it cannot be expected that numerical and analytical match in this case. In fact the drawdown is higher for the 2D numerical result, representing the unconfined situation. This is expected, as the available filter length is smaller in the unconfined case, more resembling the not ideal wells treated before.

As for the previous cases for the partially penetrating cases the drawdown is significantly higher than for the fully penetrating long screen. If the well screen measures a quarter of the aquifer thickness, the drawdown within the well is again approximately 4 times higher than for the ideal well. As in the previous cases the analytical solution is in no way representing the drawdown anymore, if the well is not ideal.



**Figure 5.** Drawdown as function of time in the pumping well and at two observation points; 2D numerical solution for the unconfined case in colour with markers, 1D analytical solution for the confined case (Theis) indicated by grey markers

## 4. Conclusions

A numerical model is set up using COMSOL Multiphysics, simulating unsteady drawdown in an aquifer due to pumping. The model uses 2D cylindrical coordinates for modelling a vertical cross-section. The 2D results are compared with the 1D analytical solutions that are commonly used in pumping test evaluations. Numerical and analytical solutions coincide perfectly, if the conditions for the validity of the analytical solutions are fulfilled. As the analytical solutions are derived for fully penetrating wells in confined or leaky aquifers, they are not valid for partially penetrating wells or unconfined aquifers. The 1D analytical solution cannot represent the 2D flow patterns in the vicinity of the well. In contrast the numerical solutions can do the job. The model runs show clearly that the 1D solutions become invalid, when the well is not ideal. If the well screen is cut by a factor of half, the drawdown increases roughly by a factor of 2, indicating inverse proportionality.

The presented numerical approach for confined and leaky aquifers is extended to simulate drawdown due to pumping in unconfined aquifers. In the model the water table is then conceived as a free boundary. The exact position of the free boundary is a result of the computations. In COMSOL Multiphysics this is performed using the moving mesh ALE mode. Results of the 2D models show how much the drawdown calculated for the unconfined case deviates from the confined case.

Concerning pumping tests the proposed modelling approach can account for

- fully and partially penetrating wells
- confined, leaky and unconfined aquifers
- anisotropy
- various far field boundary conditions
- skin-zone effects (as in Huang *et al.* 2015)
- constant rate or constant head pumping tests (as in Chang *et al.* 2011)
- changing pumping rates

Due to the paper limitations the focus here lied on the first two items. It can be concluded that COMSOL Multiphysics offers all options to model pumping tests taking various specific and local circumstances into account. In connection with an optimization module the proposed model approach can be used to evaluate pumping tests for general situations, not restricted to idealized conditions as in most commonly applied evaluation tools. For the optimization one may use a MATLAB module or the COMSOL Multiphysics toolbox concerning this matter. Development in this direction is in progress.

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