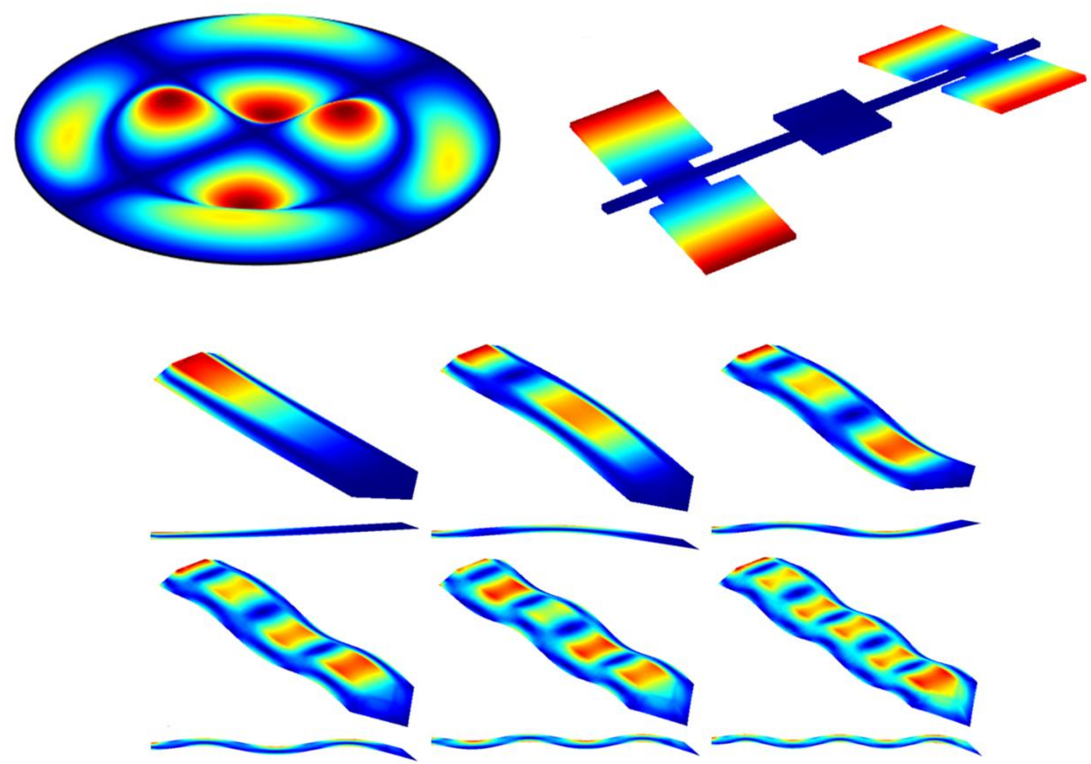
Effective Mass Calculations Using COMSOL Multiphysics® for Thermomechanical Calibration

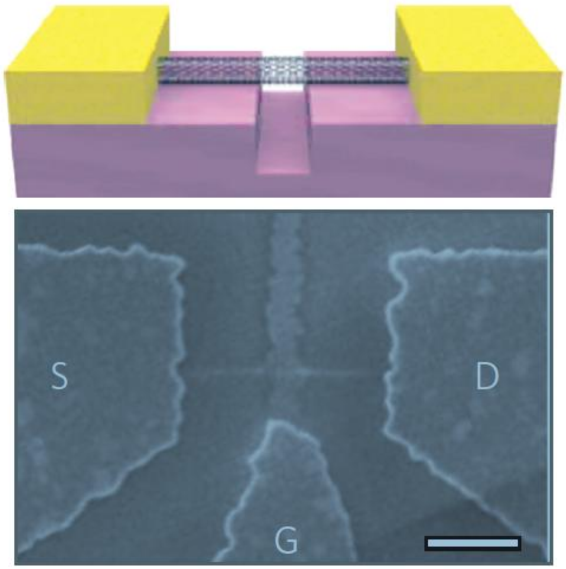
COMSOL CONFERENCE
BOSTON 2013



B.D. Hauer, C. Doolin, K.S.D. Beach, and J.P. Davis, Annals of Physics **339,** 181 (2013)

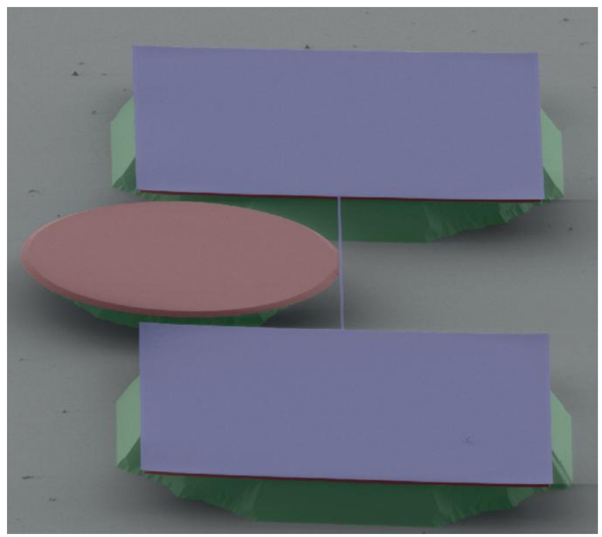
Nano/Micro-Mechanical Sensors

Yoctogram (10^{-24} g) mass resolution



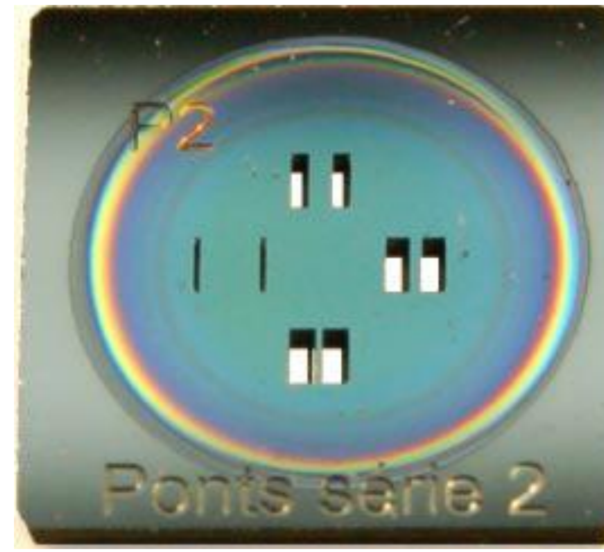
J. Chaste *et al.*, Nature Nanotech. **7** (2012) 301

Attonewton (10^{-18} N) force transduction



E. Gavartin *et al.*, Nature Nanotech. **7** (2012) 509

Sub-attometer (10^{-19} m) displacement sensitivity

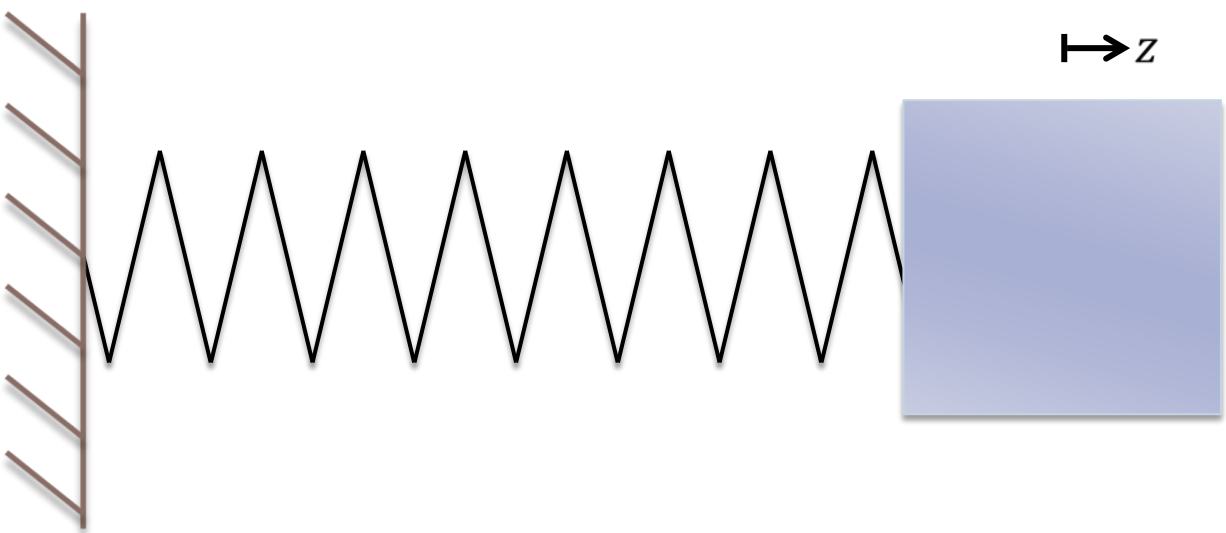


O. Arcizet *et al.*, PRL **97** (2006) 133601

Thermomechanical Calibration

Proper calibration of a resonator is crucial to ensure accurate measurements. Thermomechanical calibrations provides a powerful, noninvasive calibration by which the thermal motion of any resonator structure can be calibrated.

Equipartition Theorem: $\langle U \rangle = \frac{1}{2} m\omega^2 \langle z^2 \rangle = \frac{1}{2} k_B T$



Extended Resonator Structure

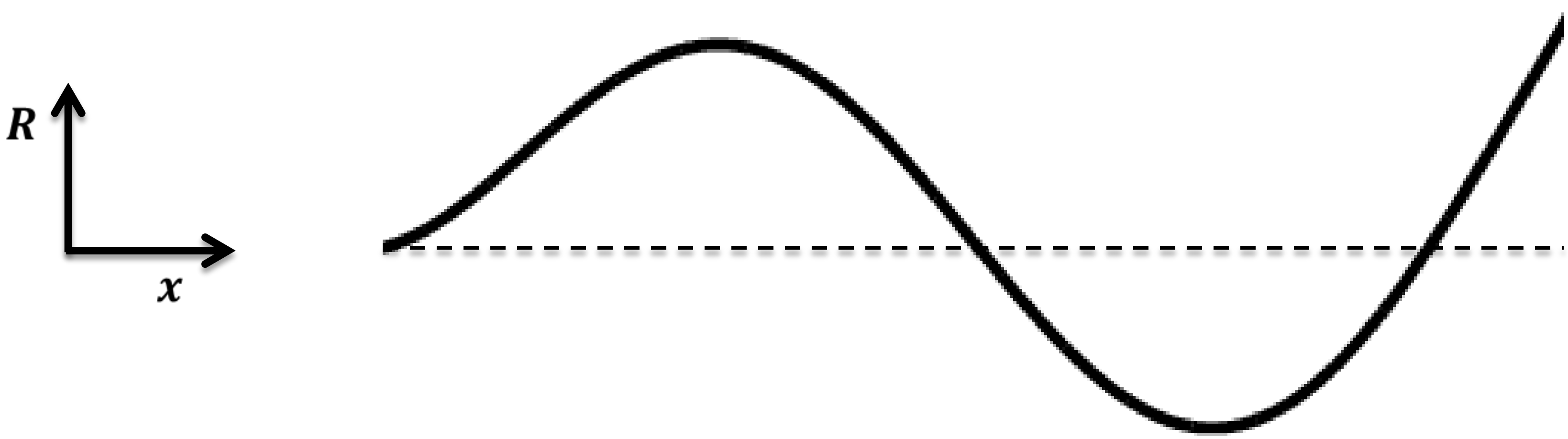
General Displacement

$$\boldsymbol{R}(\boldsymbol{x}, t) = \sum_n a_n(t)\, \boldsymbol{r}_n(\boldsymbol{x})$$

$|\boldsymbol{r}_n(\boldsymbol{x}_0)| = 1$

$\boldsymbol{x}_0$ is the position of the measurement

$a_n(t)$ is the true physical displacement



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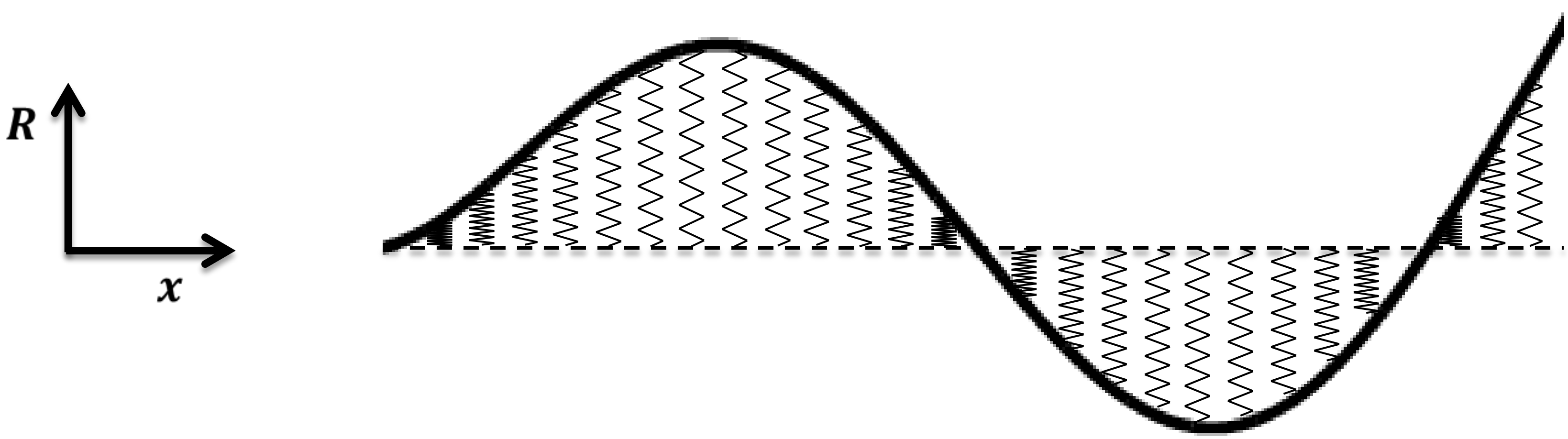
Potential Energy

$$dU = \frac{1}{2}\omega^2|a_n(t)\boldsymbol{r}_n(\boldsymbol{x})|^2\rho(\boldsymbol{x})dV \quad \Rightarrow \quad U = \frac{1}{2}m_{eff}\omega^2|a_n(t)|^2$$



Effective Mass Integral (EMI)

$$m_{eff} = \int \rho(\boldsymbol{x})|\boldsymbol{r}_n(\boldsymbol{x})|^2 dV$$

Device Geometry	Effective Mass Ratio (m_{eff} / m)
Cantilever	1/4
Doubly Clamped Beam	Mode Dependent
String	1/2
Simple Torsional Resonator	1/3
Circular Membrane	Mode Dependent
Rectangular Membrane	1/4

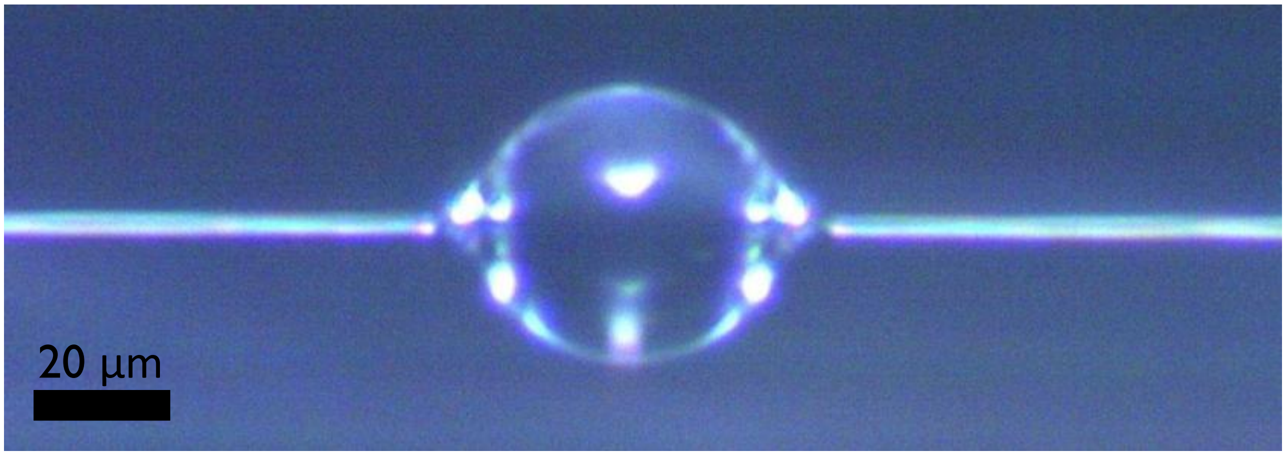
Effective Mass Integral (EMI)

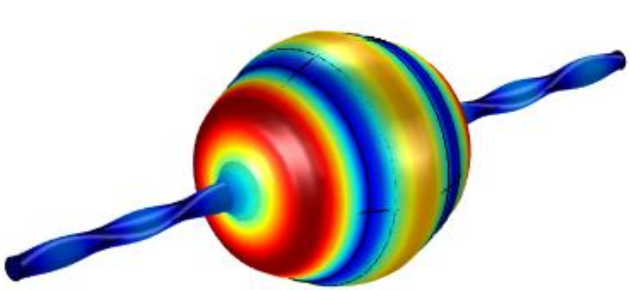
$$m_{eff} = \int \rho(\boldsymbol{x})|\boldsymbol{r}_n(\boldsymbol{x})|^2 dV$$

Device Geometry	Effective Mass Ratio (m_{eff} / m)
Cantilever	1/4
Doubly Clamped Beam	0.3965 0.4390 0.4371
String	1/2
Simple Torsional Resonator	1/3
Circular Membrane	0.2695 0.2396 0.2437
Rectangular Membrane	1/4

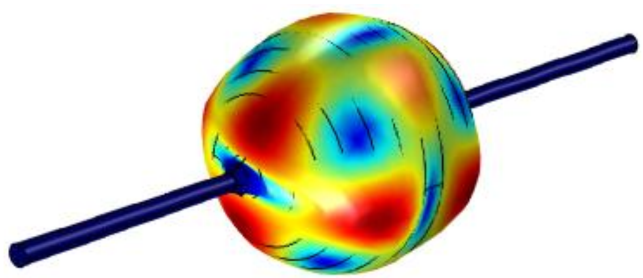
Complex Devices

Bottle Resonator

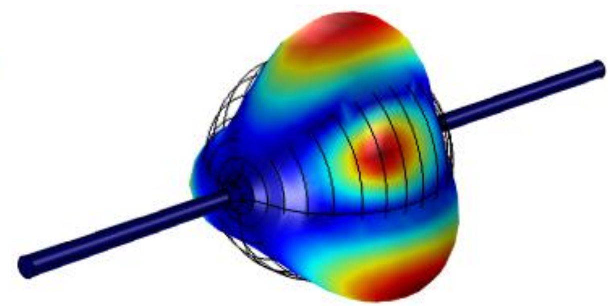




f_0 = 96.3 MHz



f_0 = 99.7 MHz



f_0 = 102.4 MHz

EMI Calculations Using COMSOL

Recipe:

1. Simulate the mechanical deformation of the device for the mode of interest using the Eigenfrequency Study in the Structural Mechanics Module.
2. Determine the relative displacement at the point of measurement using Point Evaluation in Derived Values.
3. Perform a Volume Integration of the structure's density multiplied by the normalized displacement

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solid.rho*(solid.disp/pointdisplacement))^2
```

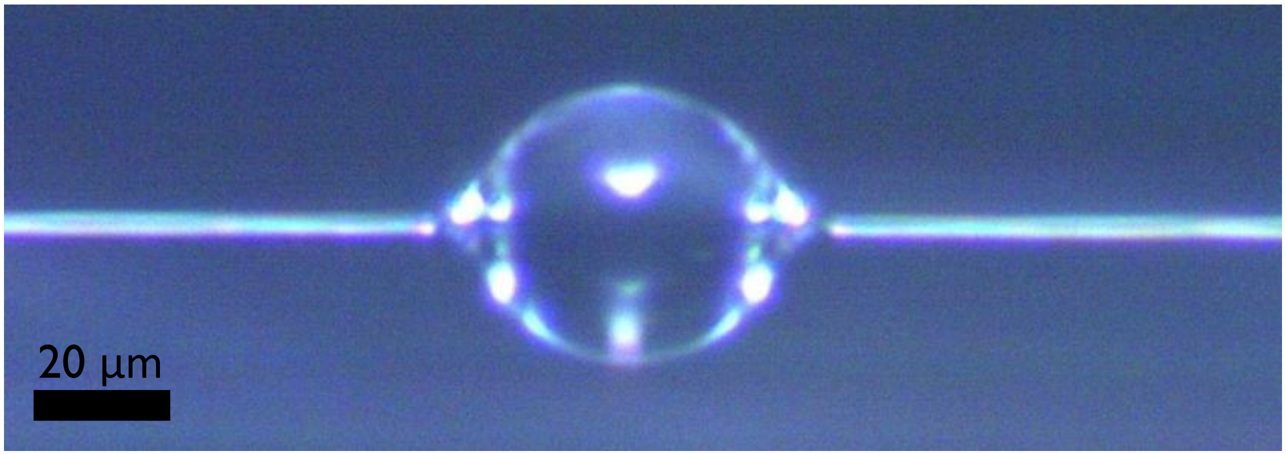
   over the entire geometry of the resonator.

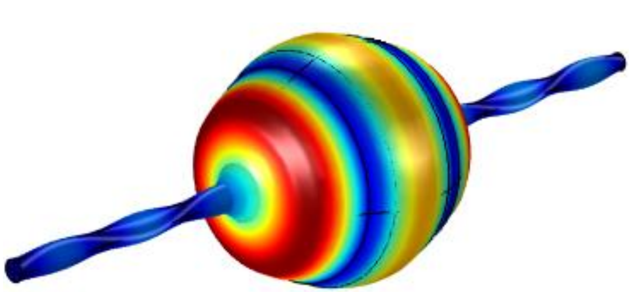
Results – Benchmark Calculations

Device Geometry	Effective Mass Ratio (Analytical)	Effective Mass Ratio (COMSOL)
Cantilever	0.2500	0.2498 (0.08%)
String	0.5000	0.4969 (0.62%)
Torsional Resonator	0.3333	0.3314 (0.58%)
Doubly Clamped Beam (First Mode)	0.3965	0.3959 (0.15%)
Circular Membrane	0.2696	0.2693 (0.11%)
Rectangular Membrane (First Mode)	0.2500	0.2498 (0.08%)

Complex Devices

Bottle Resonator





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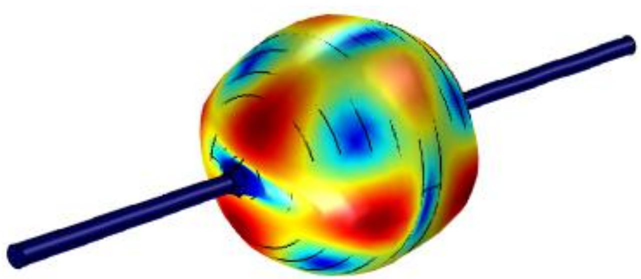

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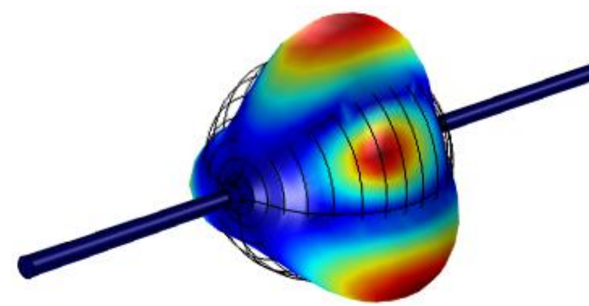
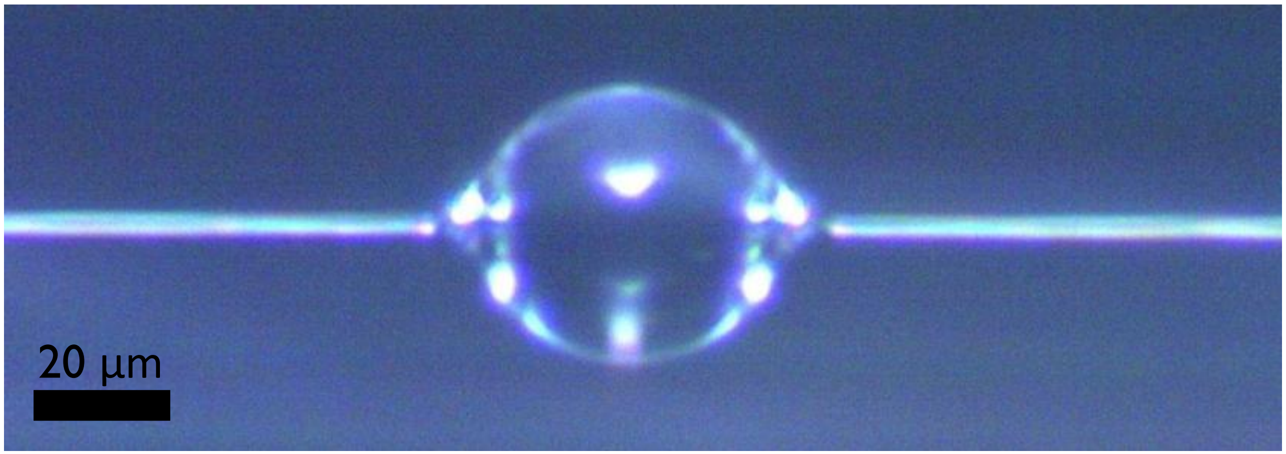
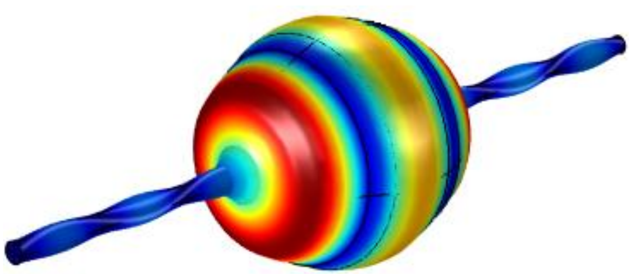


f_0 = 102.4 MHz

Complex Devices

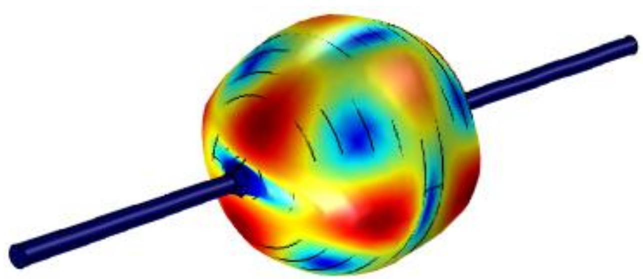
Bottle Resonator





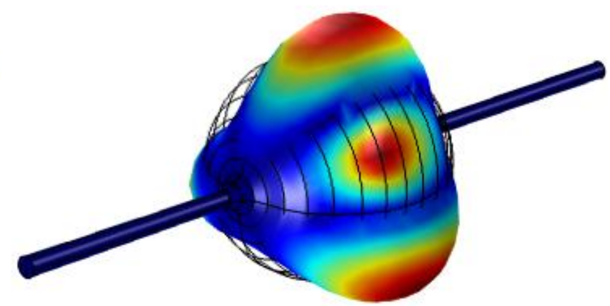
f_0 = 96.3 MHz

m_{eff}/m = 0.1890



f_0 = 99.7 MHz

m_{eff}/m = 0.1928



f_0 = 102.4 MHz

m_{eff}/m = 0.1506

Conclusion

- Thermomechanical calibration provides a powerful, non-invasive method by which nano/micro-mechanical resonators can be calibrated.
- In order to ensure accurate calibration, the effective mass of a device must be precisely known.
- COMSOL Multiphysics® simulation can be used to determine the effective mass of any resonator, regardless of material or geometry, providing a straightforward method by which any nano/micromechanical resonator can be calibrated.

B.D. Hauer, C. Doolin, K.S.D. Beach, and J.P. Davis, Annals of Physics **339,** 181 (2013)

Thank you