Models with Helical Symmetry Studied in a 2D Plane

Martijn Weterings
Michael Beyrer

HES-SO Valais-Wallis • rte du Rawyl 47 • C.P. • 1950 Sion 2
+41 27 606 86 11 • info.itv@hevs.ch • **www.hevs.ch**

Motivation: Partial differential equations (PDEs) in multiple dimensions may often be solved in a lower dimension if the problem domain contains a symmetry (cylindrical, spherical, translational, etcetera).

- In the case of a helical symmetry, the common strategy to solve PDEs in a lower dimension is a transformation into helical coordinates. This increases the complexity of equations and it has only been used for problems with PDEs of little complexity (Wang 2006, Pozrikidis 2006).
- Helical symmetric problems with complicated PDEs, like turbulent flow, are not solved using of the helical symmetry but make use of a periodic symmetry only. This is done by studying a small piece of the domain and applying periodic boundary conditions (Kim 2000, Pisarenco 2009).

Proposal: For problems involving PDEs with helical symmetry we propose a method, with both low computational costs and low complexity, that reduces the domain to a 2D axial plane while retaining 3D Cartesian coordinates.

- By dropping the axial direction from the domain, the derivative in this direction is not computed.
- The equations are made complete by rewriting the derivative in axial direction in terms of derivatives in the directions tangent to the axial plane.
- This is only a minimal transformation compared to a full transformation into helical coordinates.

Substitution rules for the partial derivative in axial direction: A substitution rule for the partial derivative in axial direction is derived using the equivalence relation that relates the solution in the entire domain (a vector, or other dependant variable v^α in the point x^β) to the solution in the axial plane at $x^c = 0$ (v^δ in the point x^γ).

Figure 1 displays an intuitive view of the relation from which the substitution is derived. The relation is based on rotations of coordinates, S^γ_β, and vectors, T^α_δ. Using the Cartesian coordinates x^a, x^b, x^c and the Einstein summation convention for Greek letters, the equivalence relation from which to derive the substitution can be expressed as:

$$v^\alpha(x^\beta) = T^\alpha_\delta v^\delta(S^\gamma_\beta x^\beta)$$
$$\text{with} \quad T^\alpha_\delta = \begin{pmatrix} cos(\tau x^c) & -sin(\tau x^c) & 0 \\ sin(\tau x^c) & cos(\tau x^c) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S^\beta_\gamma = \begin{pmatrix} cos(\tau x^c) & sin(\tau x^c) & 0 \\ -sin(\tau x^c) & cos(\tau x^c) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$

Derivation of equation 1 (where $v^i_{,j}$ is the j-th partial derivative of the i-th vector component), applying Leibniz's product rule and chain rule, and evaluating in the plane $x^c = 0$ such that $cos(\tau x^c) = 1$ and $sin(\tau x^c) = 0$, leads to the following substitution equations for derivatives of the coordinate x^c:

$$\begin{aligned}
v^a_{,c} &= \tau(v^a_{,a}x^b - v^a_{,b}x^a - v^b), & v^a_{,bc} &= \tau(v^a_{,ba}x^b - v^a_{,bb}x^a + v^a_{,a} - v^b_{,b}), & v^a_{,ac} &= \tau(v^a_{,aa}x^b - v^a_{,ab}x^a - v^a_{,b} - v^b_{,a}) \\
v^b_{,c} &= \tau(v^b_{,a}x^b - v^b_{,b}x^a + v^a), & v^b_{,bc} &= \tau(v^b_{,ba}x^b - v^b_{,bb}x^a + v^b_{,a} + v^a_{,b}), & v^b_{,ac} &= \tau(v^b_{,aa}x^b - v^b_{,ab}x^a - v^b_{,b} + v^a_{,a}) \\
v^c_{,c} &= \tau(v^c_{,a}x^b - v^c_{,b}x^a), & v^c_{,bc} &= \tau(v^c_{,ba}x^b - v^c_{,bb}x^a + v^c_{,a}), & v^c_{,ac} &= \tau(v^c_{,aa}x^b - v^c_{,ab}x^a - v^c_{,b})
\end{aligned}$$

$$\begin{aligned}
v^a_{,cc} &= \tau^2(v^a_{,bb}x^a x^a + v^a_{,aa}x^b x^b - 2v^a_{,ab}x^a x^b - v^a_{,a}x^a - v^a_{,b}x^b - 2v^b_{,a}x^b + 2v^b_{,b}x^a - v^a) \\
v^b_{,cc} &= \tau^2(v^b_{,bb}x^a x^a + v^b_{,aa}x^b x^b - 2v^b_{,ab}x^a x^b - v^b_{,a}x^a - v^b_{,b}x^b - 2v^a_{,a}x^b + 2v^a_{,b}x^a - v^b) \\
v^c_{,cc} &= \tau^2(v^c_{,bb}x^a x^a + v^c_{,aa}x^b x^b - 2v^c_{,ab}x^a x^b - v^c_{,a}x^a - v^c_{,b}x^b)
\end{aligned} \tag{2}$$

Application to corrugated tubes: As an example the substitution method is applied to the problem of turbulent flow in a helically symmetric corrugated tube.

The k-g turbulent model (Kalitzin 1997, cited in Kalitzin and Iacarrino 2002) is used to describe a stationary state of the fluid motion. In this model the second closure equation uses the parameter $g = \sqrt{\frac{1}{C_\mu \omega}}$ and reads $\partial_t g + u \cdot \nabla g = -\alpha \frac{g}{2k} P_k + \frac{\beta_1}{2gC_\mu} - (\nu + \sigma_g \nu_t) \frac{3}{g} \nabla g \cdot \nabla g + \nabla \left[(\nu + \sigma_g \nu_t) \nabla g \right]$.

Stability Computations were performed in COMSOL using the weak form PDE module. The stability of convergence was improved by gradually refining the mesh and increasing the pressure gradient (see Figure 2). Other keywords in improving the stability of convergence for this case: iteration damping factor, discretization, pseudo time stepping and pivoting perturbation.

Results A large range of values and parameters can be swept due to the low computational costs. Figure 3 displays results of such a sweep study in a pressure chart. Large variations in the friction factor, depending on the parameters, occur in the turbulent transition zone. In the turbulent region the helical tubes have a higher friction factor than tubes with similar relative roughness.

Discussion This method is very efficient and appears to be promising for further analysis of a full range of dimensionless parameters describing the corrugation geometry as well as studying differences of corrugations shapes. It is also desirable to include heat transfer into the two dimensional analysis but this may be more difficult since the temperature distribution along a tube is not constant and hence the solution has no helical symmetry.

References

G. Kalitzin and D. Iaccarino, *Turbulence modeling in an immersed-boundary RANS method.* CTR Annual Research Briefs, 415-426, 2002

J-H. Kim. *Fin shape effects in turbulent heat transfer in tubes with helical fins.* Phd Thesis Rensselaer Polytech Institute New York, 2000

B. J. McKeon, M. V. Zagarola, A. J. Smits. *A new friction factor relationship for fully developed pipe flow* J. Fluid Mech. 538:429-443, 2005

M. Pisarenco. *Friction factor estimation for turbulent flows in corrugated pipes with rough walls.* Msc Thesis TU Eindhoven, 2007

C. Pozrikidis. *Stokes flow through a twisted tube.* J fluid Mech, 567:261–280, 2006

C. Y. Wang. *Effect of helical corrugations on the low reynolds number flow in a tube.* AiChe J, 52(4):2008–2012, 2006

This research is part of CTI funded project no. 14936.1

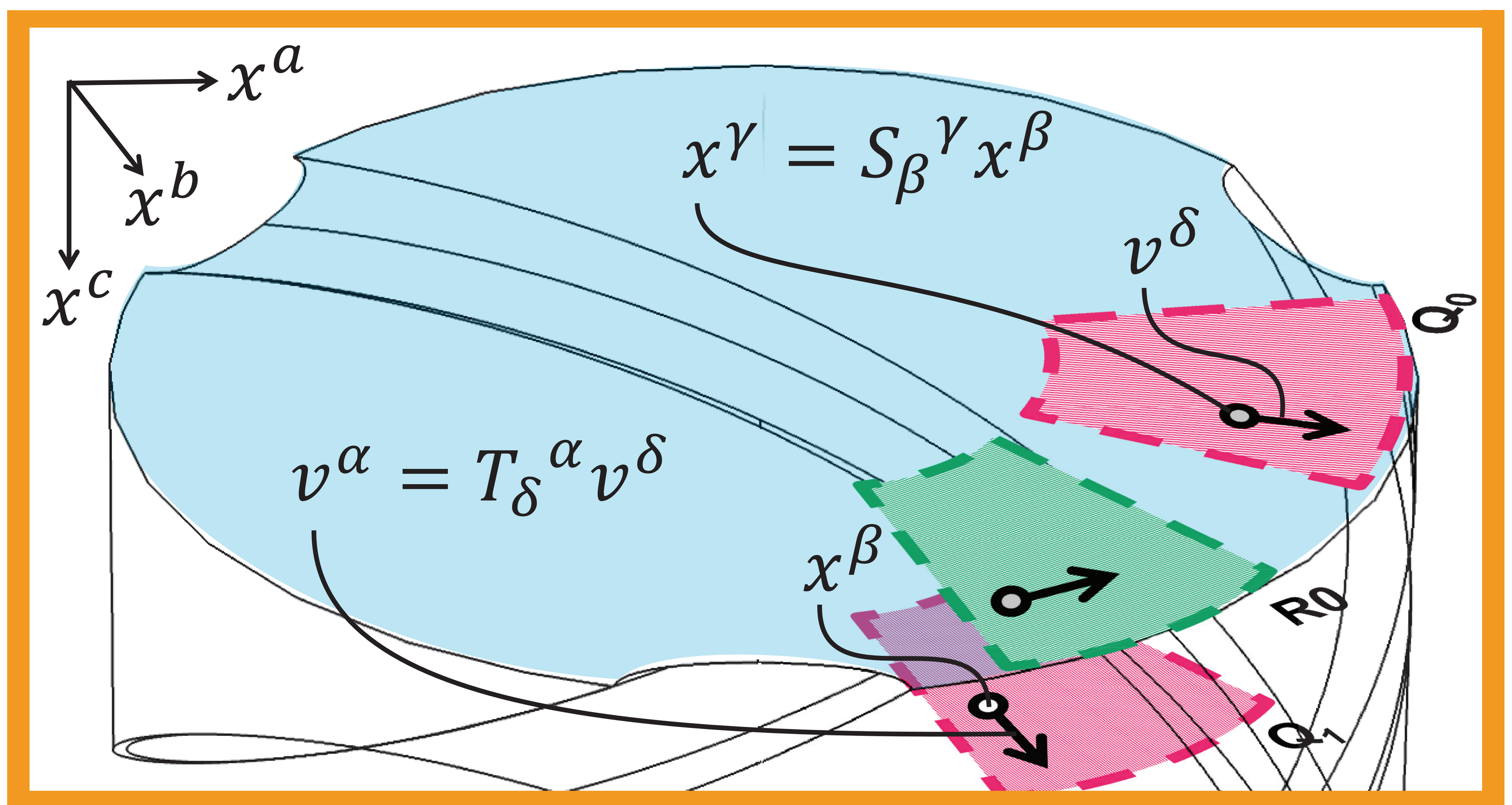

Figure 1: Solving in the axial plane (light blue circle)... an intuitive view
- The solutions in the two regions Q_0 and Q_1 are equivalent under rotation.
- This indicates that the derivative in the axial direction can be expressed in terms of derivatives in the axial plane (change from R_0 to Q_0 and R_0 to Q_1 are equivalent).
- The expression can be found by derivation of the equivalence relation in Eq. 1.

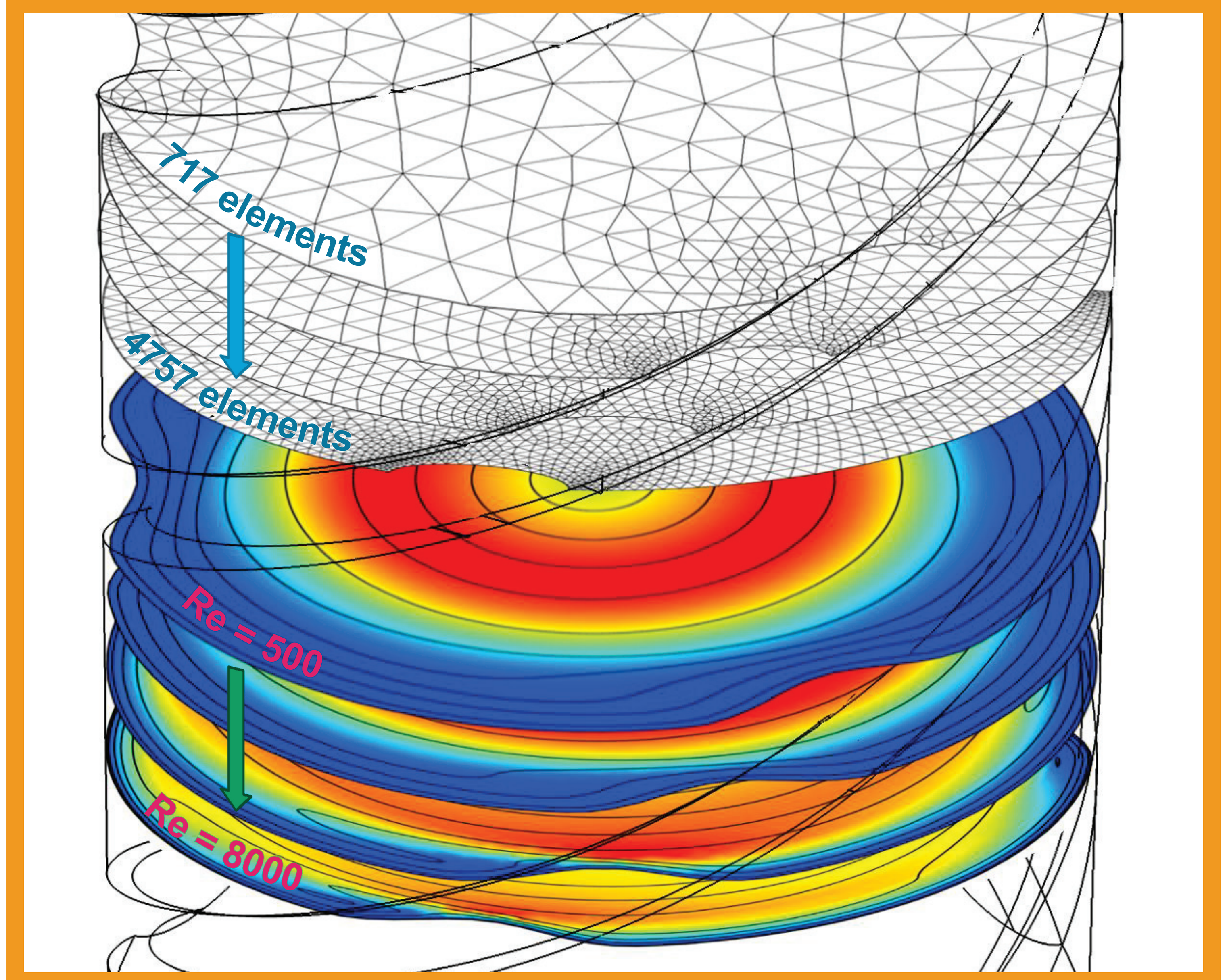

Figure 2: Convergence at high Reynolds numbers and small mesh size
Low quality of initial values may lead to bad convergence. To improve convergence the CFD problem is first solved with a small pressure gradient while the mesh is gradually refined. In a second step the pressure gradient is gradually increased. The plotted values in the display are k (rainbow colours), and g (contour lines).

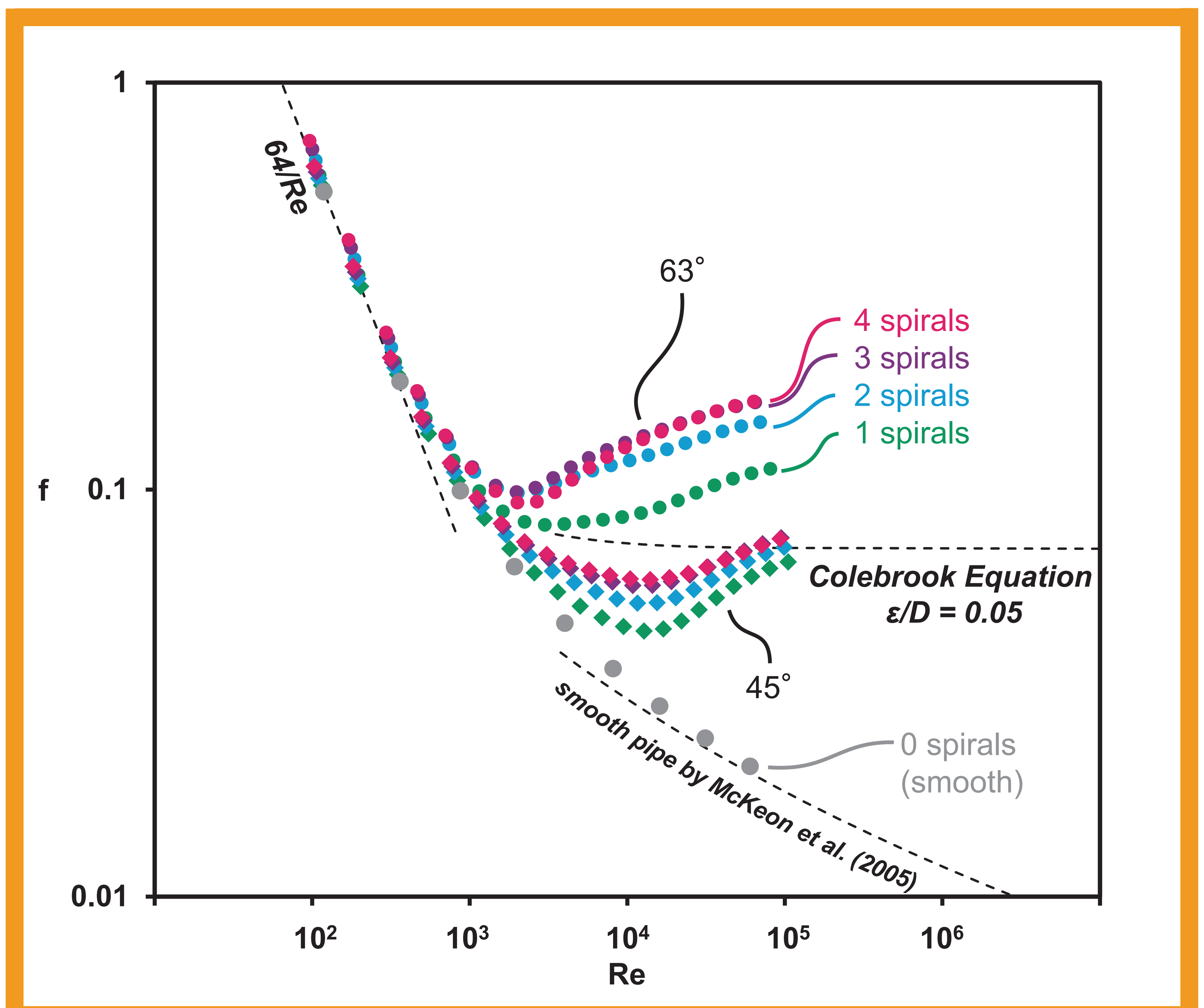

Figure 3: A parametric sweep study varying pitch and number of corrugations.
The pitch is $\tau = 1$ ($45°$) for squares and $\tau = 2$ ($63°$) for circles. The number of spirals ranges from 1 to 4 (color coded). Also calculated is a smooth tube (grey dots).

Excerpt from the Proceedings of the 2013 COMSOL Conference in Rotterdam